

E U C L I D's
ELEMENTS
OF
GEOMETRY,

FROM THE
Latin Translation of COMMANDINE.

To which is added,
A TREATISE of the *Nature* of *Arithmetic* of
LOGARITHMS; Likewise
Another of the ELEMENTS of *Plain* and *Spherical*
TRIGONOMETRY; With
A PREFACE, shewing the *Usefulness* and *Excellency* of
this WORK.

By Doctor JOHN KEIL, F. R. S. and late
Professor of ASTRONOMY in *Oxford*.

The *Whole* Revised; where deficient, Supplied; where lost
or corrupted, Restored. Also

Many Faults committed by Dr. HARRIS, Mr. CASWEL,
Mr. HEYNES, and other TRIGONOMETRICAL
WRITERS, are shewn, and in those Cases
where *They* are mistaken, here are
given Solutions *Geometrically* True.

An Ample Account of which may be seen in the PREFACE,
By SAMUEL CUNN.

THE FIFTH EDITION.

To which is subjoined an APPENDIX, containing the
Investigation of those Series's omitted by the AUTHOR.
And the Difference between Dr. KEIL and Mr.
CUNN impartially examined and adjusted.

L O N D O N:

Printed for T. WOODWARD; and Sold by J. OSBORN,
at the *Golden Ball* in *Pater-noster Row*.
M.DCC.XLV.

EUCLID'S ELEMENTS OF GEOMETRY



Latin Translation of COMMANDINE.

A TREATISE of the ELEMENTS of GEOMETRY.
Another of the ELEMENTS of ARITHMETIC.
A TREATISE of the ELEMENTS of ASTRONOMY.
A TREATISE of the ELEMENTS of COSMOGRAPHY.

By E. J. BOYD, F. R. S. and
Fellow of the Astronomical Society.

The Author's Name, as printed, is
of Cambridge, England.

Now published by J. H. B. and
J. H. B. and J. H. B. and J. H. B.

W. J. B. and J. H. B. and J. H. B.

At the University of Cambridge.

THE FIRST EDITION.

To which is added an APPENDIX, containing the

last part of the series, omitted in the first.

And the Difference between the Latin and
Greek Editions, examined and reported.

ALSO BY THE SAME AUTHOR,

A TREATISE of the ELEMENTS of ASTRONOMY.

Dr. KEIL's PREFACE.

A YOUNG Mathematician may be surpris'd, to see the old obsolete Elements of Euclid appear afresh in Print; and that too after so many new Elements of Geometry, as have been lately publish'd; especially since those who gave us the Elements of Geometry, in a new manner, would have us believe they have detected a great many Faults in Euclid. These acute Philosophers pretend to have discovered, that Euclid's Definitions are not perspicuous enough; that his Demonstrations are scarcely evident; that his whole Elements are ill-dispos'd; and that they have found out innumerable Falsities in them, which had lain hid to their Times.

But, by their Leave, I make bold to affirm, that they carp at Euclid undeservedly: For his Definitions are distinct and clear, as being taken from first Principles, and our most easy and simple Conceptions; and his Demonstrations elegant, perspicuous, and concise, carrying with them such Evidence, and so much Strength of Reason, that I am easily induced to believe the Obscurity Sciolists so often accuse Euclid with, is rather to be attributed to their own perplexed Ideas,

A 2 than

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than to the Demonstrations themselves. And however some may find Fault with the Disposition and Order of his Elements, yet notwithstanding I do not find any Method, in all the Writings of this kind, more proper and easy for Learners than that of Euclid.

It is not my Business here to answer separately every one of these Cavillers; but it will easily appear to any one, moderately versed in these Elements, that they rather shew their own Idleness, than any real Faults in Euclid. Nay, I dare venture to say, there is not one of these new Systems, wherein there are not more Faults, nay, grosser Paralogisms, than they have been able even to imagine in Euclid.

After so many unsuccessful Endeavours, in the Reformation of Geometry, some very good Geometricians, not daring to make new Elements, have deservedly prefer'd Euclid to all others; and have accordingly made it their Business to publish those of Euclid. But they, for what Reason I know not, have intirely omitted some Propositions, and have altered the Demonstrations of others for worse. Among whom are chiefly Tacquet and Dechalles, both of which have unhappily rejected some elegant Propositions in the Elements, (which ought to have been retained) as imagining them trifling and useless; such, for Example, as Prop. 27, 28, and 29, of the sixth Book, and some others, whose Uses they might not know. Farther, where-ever they use Demonstrations of their own,



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own, instead of Euclid's, in those Demonstrations they are faulty in their Reasoning, and deviate very much from the Conciseness of the Antients.

In the fifth Book, they have wholly rejected Euclid's Demonstrations, and have given a Definition of Proportion different from Euclid's; and which comprehends but one of the two Species of Proportion, taking in only commensurable Quantities. Which great Fault no Logician or Geometrician would have ever pardoned, had not those Authors done laudable Things in their other Mathematical Writings. Indeed, this Fault of theirs is common to all Modern Writers of Elements, who all split on the same Rock; and, to shew their Skill, blame Euclid, for what, on the contrary, he ought to be commended; I mean the Definition of proportional Quantities, wherein he shews an easy Property of those Quantities, taking in both commensurable and incommensurable ones, and from which all the other Properties of Proportionals do easily follow.

Some Geometricians, forsooth, want a Demonstration of this Property in Euclid; and undertake to supply the Deficiency by one of their own. Here, again, they shew their Skill in Logic, in requiring a Demonstration for the Definition of a Term; that Definition of Euclid being such as determines those Quantities Proportionals, which have the Conditions specified in the said Definition. And why might not the Author of

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the Elements give what Names he thought fit to Quantities having such Requisites? Surely he might use his own Liberty, and accordingly has called them Proportionals.

But it may be proper here to examine the Method whereby they endeavour to demonstrate that Property: Which is by first assuming a certain Affection, agreeing only to one kind of Proportionals, viz. Commensurables; and thence, by a long Circuit, and a perplexed Series of Conclusions, do deduce that universal Property of Proportionals which Euclid affirms; a Procedure foreign enough to the just Methods and Rules of Reasoning. They would certainly have done much better, if they had first laid down that universal Property by Euclid, and thence have deduced that particular Property agreeing to only one Species of Proportionals. But, rejecting this Method, they have taken the Liberty of adding their Demonstration to this Definition of the fifth Book. Those who have a mind to see a farther Defence of Euclid, may consult the Mathematical Lectures of the learned Dr. Barrow.

As I have happened to mention this great Geometrician, I must not pass by the Elements published by him, wherein generally he has retained the Constructions and Demonstrations of Euclid himself, not having omitted so much as one Proposition. Hence, his Demonstrations become more strong and nervous, his Constructions more neat and elegant, and the Genius of the antient Geometricians

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metricians more conspicuous, than is usually found in other Books of this kind. To this he has added several Corollaries and Scholia, which serve not only to shorten the Demonstrations of what follows, but are likewise of Use in other Matters.

Notwithstanding this, Barrow's Demonstrations are so very short, and are involved in so many Notes and Symbols, that they are rendered obscure and difficult to one not versed in Geometry. There many Propositions, which appear conspicuous in reading Euclid himself, are made knotty, and scarcely intelligible to Learners by this Algebraical Way of Demonstration. As is, for Example, Prop. 13. Book I. and the Demonstrations which he lays down in Book II. are still more difficult: Euclid himself has done much better, in shewing their Evidence by the Contemplations of Figures, as in Geometry should always be done. The Elements of all Sciences ought to be handled after the most simple Method, and not to be involv'd in Symbols, Notes, or obscure Principles, taken elsewhere.

As Barrow's Elements are too short, so are those of Clavius too prolix, abounding in superfluous Scholiums and Comments: For in my Opinion Euclid is not so obscure as to want such a Lumber of Notes, neither do I doubt but a Learner will find Euclid himself easier than any of his Commentators. As too much Brevity in Geometrical Demonstrations begets Obscurity, so too much Prolixity produces Tedioufness and Confusion.

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On these Accounts, principally, it was, that I undertook to publish the first six Books of Euclid, with the 11th and 12th, according to Commandinus's Edition; the rest I forbore, because those first-mentioned are sufficient for understanding of most Parts of the Mathematics now studied.

Farther, for the Use of those who are desirous to apply the Elements of Geometry to Uses in Life, we have added a Compendium of Plain and Spherical Trigonometry, by means whereof Geometrical Magnitudes are measur'd, and their Dimension expressed in Numbers.



J. KEIL.

Mr. C U N N's P R E F A C E,

S H E W I N G

The U S E F U L N E S S *and* E X C E L L E N C Y
of this W O R K.

DR. *KEIL*, in his Preface, hath sufficiently declared how much easier, plainer, and more elegant, the Elements of Geometry written by *Euclid* are, than those written by others; and that the Elements themselves are fitter for a Learner, than those published by such as have pretended to comment on, symbolize, or transpose any of his Demonstrations of such Propositions as they intended to treat of. Then how must a Geometrician be amazed, when he meets with a Tract * of the 1st, 2d, 3d, 4th, 5th, 6th, 11th and 12th *Books* of the Elements, in which are omitted the Demonstrations of all the Propositions of that most noble universal Mathesis, the 5th; on which the 6th, 11th, and 12th so much depend, that the Demonstration of not so much as one Proposition in them can be obtained without those in the fifth!

* *Vide* the last Edition of the *English* Tacquet.

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The 7th, 8th, and 9th Books treat of such Properties of Numbers which are necessary for the Demonstrations of the 10th, which treats of Incommensurables; and the 13th, 14th, and 15th, of the five *Platonic* Bodies. But though the Doctrine of Incommensurables, because expounded in one and the same Plane, as the first six Elements were, claimed, by a Right of Order, to be handled before Planes intersected by Planes, or the more compounded Doctrine of Solids; and the Properties of Numbers were necessary to the Reasoning about Incommensurables; yet because only one Proposition of these four Books, *viz.* the 1st of the 10th, is quoted in the 11th and 12th Books; and that only once, *viz.* in the Demonstration of the 2d of the 12th; and that 1st Proposition of the 10th is supplied by a *Lemma* in the 12th; and because the 7th, 8th, 9th, 10th, 13th, 14th, 15th Books have not been thought (by our greatest Masters) necessary to be read by such as design to make Natural Philosophy their Study, or by such as would apply Geometry to practical Affairs, Dr. *Keil*, in his Edition, gave us only these eight Books, *viz.* the first six, and the 11th and 12th.

And as he found there was wanting a Treatise of these Parts of the Elements, as they were written by *Euclid* himself; he published his Edition without omitting any of *Euclid's* Demonstrations, except two; one of which was a second Demonstration of the 9th Proposition of the third Book; and the other a Demonstration of that Property of Proportionals called *Conversion* (contained in a *Corollary* to the

19th

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19th Proposition of the fifth Book); where, instead of *Euclid's* Demonstration, which is universal, most Authors have given us only particular ones of their own. The first of these, which was omitted, is here supplied: And that which was corrupted is here restored*.

And since several Persons, to whom the Elements of Geometry are of vast Use, either are not so sufficiently skilled in, or perhaps have not Leisure, or are not willing to take the Trouble, to read the *Latin*; and since this Treatise was not before in *English*, nor any other which may properly be said to contain the Demonstrations laid down by *Euclid* himself; I do not doubt but the Publication of this Edition will be acceptable, as well as serviceable.

Such Errors, either typographical, or in the Schemes, which were taken Notice of in the *Latin* Edition, are corrected in this.

As to the Trigonometrical Tract annexed to these Elements, I find our Author, as well as Dr. *Harris*, Mr. *Caswell*, Mr. *Heynes*, and others of the Trigonometrical Writers, is mistaken in some of the Solutions.

That the common Solution of the 12th Case of Oblique Spherics is false, I have demonstrated, and given a true one. See Page 319.

* *Vide* Page 55, 107, of *Euclid's* Works, published by Dr. *Gregory*.

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In the Solution of our 9th and 10th *Cases*, by other Authors called the 1st and 2d, where are given and sought opposite Parts, not only the afore-mentioned Authors, but all others that I have met with, have told us, that the Solutions are ambiguous; which Doctrine is, indeed, sometimes true, but sometimes false: For sometimes the *Quæsitum* is doubtful, and sometimes not; and when it is not doubtful, it is sometimes greater than 90 Degrees, and sometimes less: And sure I shall commit no Crime, if I affirm, that no Solution can be given without a just Distinction of these Varieties. For the Solution of these *Cases*, see my Directions at *Pages* 321, 322.

In the Solution of our 3d and 7th *Cases*, in other Authors reckoned the 3d and 4th, where there are given two Sides and an Angle opposite to one of them, to find the 3d Side, or the Angle opposite to it; all the Writers of Trigonometry that I have met with, who have undertaken the Solutions of these two, as well as the two following *Cases*, by letting fall a Perpendicular, which is undoubtedly the shortest and best Method for finding either of these *Quæsitæ*, have told us, that the $\left\{ \begin{array}{l} \text{Sum} \\ \text{Difference} \end{array} \right\}$ of the Vertical Angles, or Bases, shall be the sought Angle or Side, according as the Perpendicular falls $\left\{ \begin{array}{l} \text{within;} \\ \text{without;} \end{array} \right\}$ which cannot be known, unless the Species of that unknown Angle, which is opposite to a given Side, be first known.

Here

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Here they leave us first to calculate that unknown Angle, before we shall know whether we are to take the Sum or the Difference of the Vertical Angles or Bases, for the sought Angle or Base: And in the Calculation of that Angle have left us in the dark as to its Species; as appears by my Observations on the two preceding Cases.

The Truth is, the *Quæsitum* here, as well as in the two former Cases, is sometimes doubtful, and sometimes not; when doubtful, sometimes each Answer is less than 90 Degrees, sometimes each is greater; but sometimes one less, and the other greater, as in the two last-mentioned Cases. When it is not doubtful, the *Quæsitum* is sometimes greater than 90 Degrees, and sometimes less; all which Distinctions may be made without another Operation, or the Knowledge of the Species of that unknown Angle, opposite to a given Side; or, which is the same Thing, the falling of the Perpendicular within or without. For which, see my Directions at Pages 324, 325.

In the Solution of our 1st and 5th Cases, called in other Authors the 5th and 6th; where there are given two Angles, and a Side opposite to one of them, to find the 3d Angle, or the Side opposite to it; they have told us, that the $\left\{ \begin{array}{l} \text{Sum} \\ \text{Difference} \end{array} \right\}$ of the Vertical Angles, or Bases, according as the Perpendicular falls $\left\{ \begin{array}{l} \text{within,} \\ \text{without,} \end{array} \right\}$ shall be the sought Angle or Side; and that it is known whether the Perpendicular

lar

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lar falls within or without, by the Affection of the given Angles.

Here they seem to have spoken as tho' the *Quæsitum* was always determined, and never ambiguous; for they have here determined whether the Perpendicular falls within or without, and thereby whether they are to take the Sum or the Difference of the Vertical Angles or Bases for the sought Angle or Side.

But notwithstanding these imaginary Determinations, I affirm, that the *Quæsitum* here, as in the two *Cases* last-mentioned, is sometimes ambiguous, and sometimes not; and that too, whether the Perpendicular falls within, or whether it falls without. See my Solutions of these two *Cases* in *Page* 323.

The Determination of the 3d *Case* of Oblique Plain Triangles, see in *Page* 325.



SAM. CUNN.

E U C L I D's ELEMENTS.

B O O K I.

DEFINITIONS.

- I. *A POINT is that which hath no Parts or Magnitude.*
- II. *A Line is Length, without Breadth.*
- III. *The Ends (or Bounds) of a Line are Points.*
- IV. *A Right Line is that which lieth evenly between its Points.*
- V. *A Superficies is that which hath only Length and Breadth.*
- VI. *The Bounds of a Superficies are Lines.*
- VII. *A plain Superficies is that which lieth evenly between its Lines.*
- VIII. *A plain Angle is the Inclination of two Lines to one another in the same Plane, which touch each other, but do not both lie in the same Right Line.*
- IX. *If the Lines containing the Angle be Right ones, then the Angle is called a Right-lined Angle.*
- X. *When a Right Line, standing on another Right Line, makes Angles on either Side thereof,*

of, equal between themselves, each of these equal Angles is a Right one; and that Right Line which stands upon the other, is called a Perpendicular to that whereon it stands.

XI. An Obtuse Angle is that which is greater than a Right one.

XII. An Acute Angle is that which is less than a Right one.

XIII. A Term (or Bound) is that which is the Extreme of any Thing.

XIV. A Figure is that which is contained under one or more Terms.

XV. A Circle is a plain Figure, contained under one Line, called the Circumference; to which all Right Lines, drawn from a certain Point within the Figure, are equal.

XVI. And that Point is called the Centre of the Circle.

XVII. A Diameter of a Circle is a Right Line drawn through the Centre, and terminated on both Sides by the Circumference, and divides the Circle into two equal Parts.

XVIII. A Semicircle is a Figure contained under a Diameter, and that Part of the Circumference of a Circle cut off by that Diameter.

XIX. A Segment of a Circle is a Figure contained under a Right Line, and Part of the Circumference of the Circle [which is cut off by that Right Line.]

XX. Right-lined Figures are such as are contained under Right Lines.

XXI. Three-sided Figures are such as are contained under three Lines.

XXII. Four-sided Figures are such as are contained under four Lines.

XXIII. Many-sided Figures are those that are contained under more than four Right Lines.

XXIV. Of



- XXIV. Of three-sided Figures, that is an Equilateral Triangle, which hath three equal Sides.
- XXV. That an Isosceles, or Equicrural one, which hath only two Sides equal.
- XXVI. And a Scalene one, is that which hath three unequal Sides.
- XXVII. Also of three-sided Figures, that is a Right-angled Triangle, which hath a Right Angle.
- XXVIII. That an Obtuse-angled one, which hath an Obtuse Angle.
- XXIX. And that an Acute-angled one, which hath three Acute Angles.
- XXX. Of four-sided Figures, that is a Square, whose four Sides are equal, and its Angles all Right ones.
- XXXI. That an Oblong, or Rectangle, which is longer than broad; but its opposite Sides are equal, and all its Angles Right ones.
- XXXII. That a Rhombus, which hath four equal Sides, but not Right Angles.
- XXXIII. That a Rhomboides, whose opposite Sides and Angles only are equal.
- XXXIV. All Quadrilateral Figures, besides these, are called Trapezia.
- XXXV. Parallels are such Right Lines in the same Plane, which, if infinitely produc'd both ways, would never meet.

POSTULATES.

- I. **G**RANT that a Right Line may be drawn from any one Point to another.
- II. That a finite Right Line may be continued directly forwards.
- III. And that a Circle may be described about any Centre, with any Distance.

B

AXIOMS.

A X I O M S.

- I. **T**HINGS equal to one and the same Thing, are equal to one another.
- II. If to equal Things are added equal Things, the Wholes will be equal.
- III. If from equal Things equal Things be taken away, the Remainders will be equal.
- IV. If equal Things be added to unequal Things, the Wholes will be unequal.
- V. If equal Things be taken from unequal Things, the Remainders will be unequal.
- VI. Things which are double to one and the same Thing, are equal between themselves.
- VII. Things which are half one and the same Thing, are equal between themselves.
- VIII. Things which mutually agree together, are equal to one another.
- IX. The Whole is greater than its Parts.
- X. Two Right Lines do not contain a Space.
- XI. All Right Angles are equal between themselves.
- XII. If a Right Line, falling upon two other Right Lines, makes the inward Angles on the same Side thereof, both together, less than two Right Angles, those two Right Lines, infinitely produc'd, will meet each other on that Side where the Angles are less than Right ones.

Note, When there are several Angles at one Point, any one of them is express'd by three Letters, of which that at the Vertex of the Angle is plac'd in the Middle. For Example; In the Figure of *Prop.* XIII. *Lib.* I. the Angle contain'd under the Right Lines A B, B C, is called the Angle A B C; and the Angle contain'd under the Right Lines A B, B E, is called the Angle A B E.

PROPOSITION I.

PROBLEM.

To describe an Equilateral Triangle upon a given finite Right Line.

LET AB be the given finite Right Line, upon which it is required to describe an Equilateral Triangle.

About the Centre A, with the Distance AB, describe the Circle BCD * ; and about the Centre B, with the same Distance BA, describe the Circle ACE ; and from the Point C, where the two Circles cut each other, draw the Right Lines CA, CB †.

† 1 Post.

Then because A is the Centre of the Circle DBC, AC shall be equal to AB †. And because B is the Centre of the Circle CAE, BC shall be equal to BA : but CA hath been proved to be equal to AB ; therefore both CA and CB are each equal to AB. But Things equal to one and the same Thing, are equal between themselves, and consequently CA is equal to CB ; therefore the three Sides CA, AB, BC, are equal between themselves. † 15 Def.

And so the Triangle BAC is an Equilateral one, and is described upon the given finite Right Line AB ; which was to be done.

PROPOSITION II.

PROBLEM.

At a given Point, to put a Right Line equal to a Right Line given.

LET the Point given be A, and the given Right Line BC ; it is required to put a Right Line at the Point A, equal to the given Right Line BC.

- * *Post.* 1. Draw the Right Line AC from the Point A to C*,
 † 1 of this. upon it describe the Equilateral Triangle DAC †;
 † *Post.* 2. produce DA and DC directly forwards to E and G †;
 about the Centre C, with the Distance BC, describe
 * *Post.* 3. the Circle BGH*; and about the Centre D, with
 the Distance DG, describe the Circle GKL.

- Now because the Point C is the Centre of the Circle
 † *Def.* 15. BGH, BC will be equal to CG †; and because D
 is the Centre of the Circle GKL, the Whole DL
 will be equal to the Whole DG, the Parts whereof
 DA and DC are equal; therefore the Remainders
 † *Axiom* 3. AL, GC, are also equal †. But it has been demon-
 strated, that BC is equal to CG; wherefore both AL
 and BC are each of them equal to CG. But Things
 that are equal to one and the same Thing, are equal
 to one another; and therefore likewise AL is equal
 to BC.

Whence the Right Line AL is put at the given
 Point A, equal to the given Right Line BC, which
 was to be done.

PROPOSITION III.

PROBLEM.

*Two unequal Right Lines being given, to cut off a
 Part from the greater equal to the lesser.*

LET AB and C be the two unequal Right Lines
 given, the greater whereof is AB; it is required
 to cut off a Line from the greater AB equal to the
 lesser C.

- * 2 of this. Put * a Right Line AD at the Point A, equal to the
 Line C; and about the Centre A, with the Distance
 † *Post.* 3. AD, describe a Circle DEF †.

Then, because A is the Centre of the Circle DEF,
 AE is equal to AD; and so both AE and C are
 each equal to AD; wherefore AE is likewise equal

- † *Axiom* 1. to C †.

And so there is cut off from AB, the greater of two
 given Right Lines AB and C, a Line AE equal to
 the lesser Line C; which was to be done.

PROPOSITION IV.

THEOREM.

If there are two Triangles that have two Sides of the one equal to two Sides of the other, each to each, and the Angle contained by those equal Sides in one Triangle equal to the Angle contained by the correspondent Sides in the other Triangle, then the Base of one of the Triangles shall be equal to the Base of the other, the whole Triangle equal to the whole Triangle, and the remaining Angles of one equal to the remaining Angles of the other, each to each, which subtend the equal Sides.

LET the two Triangles be ABC , DEF , which have two Sides AB , AC , equal to two Sides DE , DF , each to each, that is, the Side AB equal to the Side DE , and the Side AC to DF ; and the Angle BAC equal to the Angle EDF . I say, that the Base BC is equal to the Base EF , the Triangle ABC equal to the Triangle DEF , and the remaining Angles of the one equal to the remaining Angles of the other, each to its Correspondent, subtending the equal Sides, *viz.* the Angle ABC equal to the Angle DEF , and the Angle ACB equal to the Angle DFE .

For the Triangle ABC being applied to DEF , so as the Point A may co-incide with D , and the Right Line AB with DE , then the Point B will co-incide with the Point E , because AB is equal to DE . And since AB co-incides with DE , the Right Line AC likewise will co-incide with the Right Line DF , because the Angle BAC is equal to the Angle EDF . Wherefore also C will co-incide with F , because the Right Line AC is equal to the Right Line DF . But the Point B co-incides with E , and therefore the Base BC co-incides with the Base EF . For if the Point B co-inciding with E , and C with F , the Base BC does not co-incide with the Base EF ; then two Right Lines will contain a Space, which is impossible*. * *Ax. 10.* Therefore the Base BC co-incides with the Base EF , and is equal thereto; and consequently the whole Tri-

angle ABC will co-incide with the whole Triangle DEF , and will be equal thereto; and the remaining Angles will co-incide with the remaining Angles†, and will be equal to them, viz. the Angle ABC equal to the Angle DEF , and the Angle ACB equal to the Angle DFE . Which was to be demonstrated.

PROPOSITION V.

THEOREM.

The Angles at the Base of an Isosceles Triangle are equal between themselves : And, if the equal Sides be produced, the Angles under the Base shall be equal between themselves.

LET ABC be an *Isosceles* Triangle, having the Side AB equal to the Side AC ; and let the equal Sides AB , AC , be produced directly forwards to D and E . I say, the Angle ABC is equal to the Angle ACB , and the Angle CBD equal to the Angle BCE .

For assume any Point F in the Line BD , and from A cut off the Line AG equal * to AF , and join FC , GB .

Then, because AF is equal to AG , and AB to AC , the two Right Lines FA , AC , are equal to the two Lines GA , AB ; each to each, and contain the common Angle FAG ; therefore the Base FC is equal † to the Base GB , and the Triangle AFC equal to the Triangle AGB , and the remaining Angles of the one equal to the remaining Angles of the other, each to each, subtending the equal Sides, viz. the Angle ACF equal to the Angle ABG ; and the Angle AFC , equal to the Angle AGB . And because the Whole AF is equal to the Whole AG , and the Part AB equal to the Part AC , the Remainder BF is equal to the Remainder CG . But FC ‡ has been proved to be equal to GB ; therefore the two Sides BF , FC , are equal to the two Sides CG , GB , each to each, and the Angle BFC equal to the Angle CGB ; but they have a common Base BC . Therefore also the Triangle BFC will be equal to the Triangle CGB , and the remaining Angles of the one equal

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equal to the remaining Angles of the other, each to each, which subtend the equal Sides. And so the Angle FBC is equal to the Angle GCB ; and the Angle BCF equal to the Angle CBG . Therefore, because the whole Angle ABG has been proved equal to the whole Angle ACF , and the Part CBG equal to BCF , the remaining Angle ABC will be * equal to the remaining Angle ACB ; but these are the Angles at the Base of the Triangle ABC . It hath likewise been proved, that the Angles FBC , GCB , under the Base, are equal; therefore the Angles at the Base of Isosceles Triangles are equal between themselves; and if the equal Right Lines be produced, the Angles under the Base will be also equal between themselves.

* Ax. 3.

Coroll. Hence every Equilateral Triangle is also Equiangular.

PROPOSITION VI.

THEOREM.

If two Angles of a Triangle be equal, then the Sides subtending the equal Angles will be equal between themselves.

LET ABC be a Triangle, having the Angle ABC equal to the Angle ACB . I say, the Side AB is likewise equal to the Side AC .

For if AB be not equal to AC , let one of them, as AB , be the greater, from which cut off BD equal to AC †, and join DC . Then, because BD is equal to AC , and BC is common, DBC , BC , will be equal to AC , CB , each to each, and the Angle DBC equal to the Angle ACB , from the Hypothesis; therefore the Base DC is equal † to the Base AB , and the Triangle DBC equal to the Triangle ACB , a Part to the Whole, which is absurd; therefore AB is not unequal to AC , and consequently is equal to it.

† 3 of this.

† 4 of this.

Therefore, *if two Angles of a Triangle be equal between themselves, the Sides subtending the equal Angles are likewise equal between themselves.* Which was to be demonstrated.

B 4

Coroll.

Coroll. Hence every Equiangular Triangle is also Equilateral.

PROPOSITION VII.

THEOREM.

On the same Right Line cannot be constituted two Right Lines equal to two other Right Lines, each to each, at different Points, on the same Side, and having the same Ends which the first Right Lines have.

FOR, if it be possible, let two Right Lines AD , DB , equal to two others AC , CB , each to each, be constituted at different Points C and D , towards the same Parts CD , and having the same Ends A and B which the first Right Lines have, so that CA be equal to AD , having the same End A which CA hath; and CB equal to DB , having the same End B ; and let CD be joined.

Then, because AC is equal to AD , the Angle ACD will be equal * to the Angle ADC , and consequently the Angle ADC is greater than the Angle BCD ; wherefore the Angle BDC will be much greater than the Angle BCD . Again, because CB is equal to DB , the Angle BDC will be equal to the Angle BCD ; but it has been proved to be much greater, which is impossible. Therefore *on the same Right Line cannot be constituted two Right Lines equal to two other Right Lines, each to each, at different Points, on the same Side, and having the same Ends which the first Right Lines have; which was to be demonstrated.*

PROPOSITION VIII.

THEOREM.

If two Triangles have two Sides of the one equal to two Sides of the other, each to each, and the Bases equal, then the Angles contained under the equal Sides will be equal.

LET the two Triangles be ABC , DEF , having two Sides AB , AC , equal to two Sides DE , DF , each to each, *viz.* AB equal to DE , and AC to DF ; and let the Base BC be equal to the Base EF . I say, the Angle BAC is equal to the Angle EDF .

For if the Triangle ABC be applied to the Triangle DEF , so that the Point B may co-incide with E , and the Right Line BC with EF , then the Point C will co-incide with F , because BC is equal to EF . And so, since BC co-incides with EF , BA and AC will likewise co-incide with ED and DF . For if the Base BC should co-incide with EF , and at the same time the Sides BA , AC , should not co-incide with the Sides ED , DF , but change their Position, as EG , GF , then there would be constituted on the same Right Line two Right Lines, equal to two other Right Lines, each to each, at several Points, on the same Side, having the same Ends. But this is proved to be otherwise*; therefore it is impossible for the Sides BA , AC , not to co-incide with the Sides ED , DF , if the Base BC co-incides with the Base EF ; wherefore they will co-incide, and consequently the Angle BAC will co-incide with the Angle EDF , and will be equal to it. Therefore, *if two Triangles have two Sides of the one equal to two Sides of the other, each to each, and the Bases equal, then the Angles contained under the equal Sides will be equal*; which was to be demonstrated. * 7 of this.

PROPOSITION IX.

PROBLEM.

To cut a given Right-lin'd Angle into two equal Parts.

LET BAC be a given Right-lin'd Angle, which is required to be cut into two equal Parts.

Assume any Point D in the Right Line AB , and
 * 3 of this. cut off AE from the Line AC equal to AD * ; join
 † 1 of this. DE , and thereon make † the Equilateral Triangle
 DEF , and join AF . I say, the Angle BAC is cut
 into two equal Parts by the Line AF .

For, because AD is equal to AE , and AF is com-
 mon, the two Sides DA , AF , are each equal to the
 two Sides EA , AF , and the Base DF is equal to
 † 8 of this. the Base EF ; therefore † the Angle DAF is equal to
 the Angle EAF . Wherefore a given Right-lin'd Angle
 is cut into two equal Parts; which was to be done.

PROPOSITION X.

PROBLEM.

To cut a given finite Right Line into two equal Parts.

LET AB be a given finite Right Line, required to be cut into two equal Parts.

Upon it make * an Equilateral Triangle ABC , and
 * 1 of this. bisect † the Angle ACB by the Right Line CD . I
 † 9 of this. say, the Right Line AB is bisected in the Point D .

For, because AC is equal to CB , and CD is com-
 mon, the Right Lines AC , CD , are each equal to
 the two Right Lines BC , CD , and the Angle ACD
 † 4 of this. equal to the Angle BCD ; therefore † the Base AD ,
 is equal to the Base DB . And so the Right Line
 AB is bisected in the Point D ; which was to be done.

PROPOSITION XI.

PROBLEM.

To draw a Right Line at Right Angles to a given Right Line, from a given Point in the same.

LET AB be the given Right Line, and C the given Point. It is required to draw a Right Line from the Point C at Right Angles to AB.

Assume any Point D in AC, and make CE equal * to CD, and upon DE make † the Equilateral * ^{3 of this.} Triangle FDE, and join EC. I say, the Right † ^{1 of this.} Line FC is drawn from the Point C, given in the Right Line AB at Right Angles to AB.

For, because DC is equal to CE, and FC is common, the two Lines DC, CF, are each equal to the two Lines EC, CF; and the Base DF is equal to the Base FE. Therefore * the Angle DCF is equal * ^{8 of this.} to the Angle ECF; and they are adjacent Angles. But when a Right Line, standing upon a Right Line, makes the adjacent Angles equal, each of the equal Angles is † a Right Angle; and consequently DCF, † ^{Def. 10.} FCE, are both Right Angles. Therefore the Right Line FC, &c. which was to be done.

PROPOSITION XII.

PROBLEM.

To draw a Right Line perpendicular upon a given infinite Right Line, from a Point given out of it.

LET AB be the given infinite Line, and C the Point given out of it. It is requir'd to draw a Right Line perpendicular upon the given Right Line AB, from the Point C given out of it.

Assume any Point D on the other Side of the Right Line AB; and about the Centre C, with the Distance CD, describe * a Circle EDG, bisect † EG in H, * ^{Post. 3.} and join CG, CH, CE. I say, there is drawn the † ^{10 of this.}

Per-

Perpendicular CH on the given infinite Right Line AB, from the Point C given out of it.

For, because GH is equal to HE, and HC is common, GH and HC are each equal to EH and HC, and the Base CG is equal to the Base CE. Therefore the Angle CHG is equal † to the Angle CHE; and they are adjacent Angles. But when a Right Line, standing upon another Right Line, makes the Angles equal between themselves, each of the equal Angles is a Right one *, and the said standing Right Line is call'd a Perpendicular to that which it stands on. Therefore CH is drawn *perpendicular, upon a given infinite Right Line, from a given Point out of it*; which was to be demonstrated.

† 8 of this.

* Def. 10.

PROPOSITION XIII.

THEOREM.

When a Right Line, standing upon a Right Line, makes Angles, these shall be either two Right Angles, or together equal to two Right Angles.

FOR let a Right Line AB, standing upon the Right Line CD, make the Angles CBA, ABD. I say, the Angles CBA, ABD, are either two Right Angles, or both together equal to two Right Angles.

* Def. 8. For if CBA be equal to ABD, they are * each of them Right Angles: But if not, draw † BE from the Point B, at Right Angles to CD. Therefore the Angles CBE, EBD, are two Right Angles: And because CBE is equal to both the Angles CBA, ABE, add the Angle EBD, which is common; and the two Angles CBE, EBD, together, are † equal to the three Angles CBA, ABE, EBD, together. Again, because the Angle DBA is equal to the two Angles DBE, EBA, together, add the common Angle ABC, and the two Angles DBA, ABC, are equal to the three Angles DBE, EBA, ABC, together. But it has been prov'd, that the two Angles CBE, EBD, together, are likewise equal to these three Angles: But Things that are equal to one and the same, are * equal between themselves. Therefore likewise the Angles CBE, EBD, together, are equal to the Angles

† 11 of this.

† Ax. 2.

* Ax. 1.

Angles DBA, ABC, together; but CBE, EBD, are two Right Angles. Therefore the Angles DBA, ABC, are both together equal to two Right Angles. Wherefore *when a Right Line, standing upon another Right Line, makes Angles, these shall be either two Right Angles, or together equal to two Right Angles; which was to be demonstrated.*

PROPOSITION XIV.

THEOREM.

If to any Right Line, and Point therein, two Right Lines be drawn from contrary Parts, making the adjacent Angles, both together, equal to two Right Angles, the said two Right Lines will make but one strait Line.

FOR let two Right Lines BC, BD, drawn from contrary Parts to the Point B, in any Right Line AB, make the adjacent Angles ABC, ABD, both together equal to two Right Angles. I say, BC, BD, make but one Right Line.

For if BD, CB, do not make one strait Line, let CB and BE make one.

Then, because the Right Line AB stands upon the Right Line CBE, the Angles ABC, ABE, together, will be equal * to two Right Angles. But the Angles ABC, ABD, together, are also equal to two Right Angles. Now taking away the common Angle ABC, and the remaining Angle ABE is equal to the remaining Angle ABD, the less to the greater, which is impossible. Therefore BE, BC, are not one strait Line. And in the same manner it is demonstrated, that no other Line but BD is in a strait Line with CB; wherefore CB, BD, shall be in one strait Line. Therefore, *if to any Right Line, and Point therein, two Right Lines be drawn from contrary Parts, making the adjacent Angles, both together, equal to two Right Angles, the said two Right Lines will make but one strait Line; which was to be demonstrated.*

* 13 of this.

PROPOSITION XV.

THEOREM.

If two Right Lines mutually cut each other, the opposite Angles are equal.

LET the two Right Lines AB , CD , mutually cut each other in the Point E . I say, the Angle AEC is equal to the Angle DEB ; and the Angle CEB equal to the Angle AED .

For, because the Right Line AE , standing on the Right Line CD , makes the Angles CEA , AED ;
 * 13 of this. these both together shall be equal * to two Right Angles. Again, because the Right Line DE , standing upon the Right Line AB , makes the Angles AED , DEB ; these Angles together are * equal to two Right Angles. But it has been prov'd, that the Angles CEA , AED , are likewise together equal to two Right Angles. Therefore the Angles CEA , AED , are equal to the Angles AED , DEB . Take away the common Angle AED , and the Angle remaining CEA is † equal to the Angle remaining DEB . For the same Reason the Angle CEB shall be equal to the Angle DEA . Therefore, *if two Right Lines mutually cut each other, the opposite Angles are equal*; which was to be demonstrated.

† Ax. 3.

Coroll. 1. From hence it is manifest, that two Right Lines, mutually cutting each other, make Angles at the Section equal to four Right Angles.

Coroll. 2. All the Angles constituted about the same Point are equal to four Right Angles.

PROPOSITION XVI.

THEOREM.

If one Side of any Triangle be produced, the outward Angle is greater than either of the inward opposite Angles.

LET ABC be a Triangle, and one of its Sides BC be produced to D . I say, the outward Angle ACD is greater than either of the inward Angles CBA , or BAC .

For bisect AC in E^* , and join BE , which produce to F , and make EF equal to BE . Moreover, join FC , and produce AC to G . * 10 of this.

Then, because AE is equal to EC , and BE to EF , the two Sides AE , EB , are equal to the two Sides CE , EF , each to each, and the Angle AEB † equal to the Angle FEC ; for they are opposite † 15 of this. Angles. Therefore the Base AB is † equal to the † 4 of this. Base FC ; and the Triangle AEB equal to the Triangle FEC ; and the remaining Angles of the one, equal to the remaining Angles of the other, each to each, subtending the equal Sides. Wherefore the Angle BAE is equal to the Angle ECF ; but the Angle ECD is greater than the Angle ECF ; therefore the Angle ACD is greater than the Angle BAE . After the same manner, if the Right Line BC be bisected, we demonstrate that the Angle BCG , that is, the Angle ACD , is greater than the Angle ABC . Therefore, *one Side of any Triangle being produced, the outward Angle is greater than either of the inward opposite Angles*; which was to be demonstrated.

PROPOSITION XVII.

THEOREM.

Two Angles of any Triangle together, howsoever taken, are less than two Right Angles.

LET ABC be a Triangle. I say, two Angles of it together, howsoever taken, are less than two Right Angles. For

For produce BC to D.

Then, because the outward Angle ACD of the
 * 16 of this. Triangle ABC is greater * than the inward opposite
 Angle ABC: If the common Angle ACB be added,
 the Angles ACD, ACB, together, will be greater
 than the Angles ABC, ACB, together: But ACD,
 † 13 of this. ACB, are † equal to two Right Angles. Therefore
 ABC, BCA, are less than two Right Angles. In
 the same manner we demonstrate, that the Angles
 BAC, ACB, as also CAB, ABC, are less than
 two Right Angles. Therefore *two Angles of any Tri-
 angle together, howsoever taken, are less than two Right
 Angles*; which was to be demonstrated.

PROPOSITION XVIII.

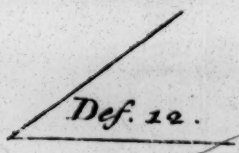
THEOREM.

*The greater Side of every Triangle subtends the
 greater Angle.*

LET ABC be a Triangle, having the Side AC
 greater than the Side AB. I say, the Angle ABC
 is greater than the Angle BCA.

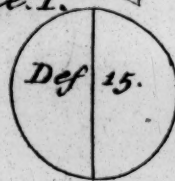
For, because AC is greater than AB, AD may be
 made equal to AB, and BD be joined.

Then, because ADB is an outward Angle of the
 * 16 of this. Triangle BDC, it will be * greater than the inward
 opposite Angle DCB. But ADB is † equal to
 † 5 of this. ABD; because the Side AB is equal to the Side AD.
 Therefore the Angle ABD is likewise greater than
 the Angle ACB; and consequently ABC shall be
 much greater than ACB. Wherefore *the greater Side
 of every Triangle subtends the greater Angle*; which
 was to be demonstrated.

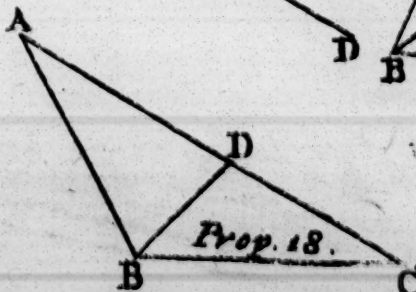
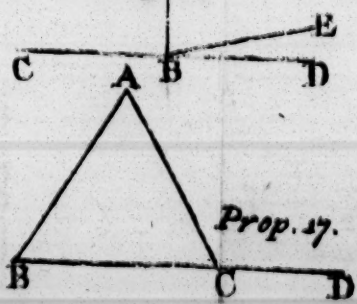
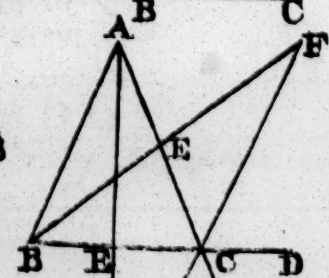
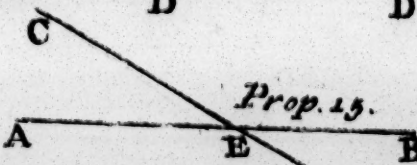
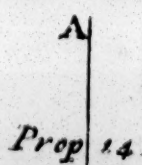
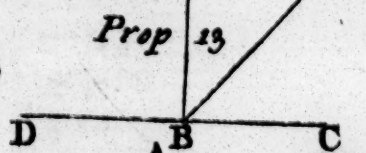
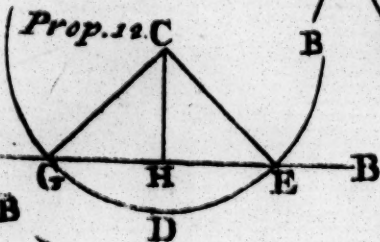
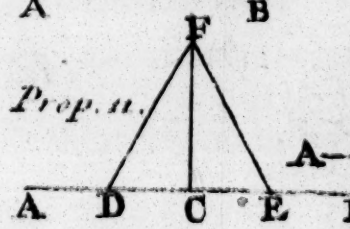
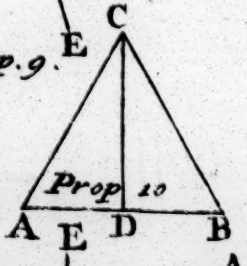
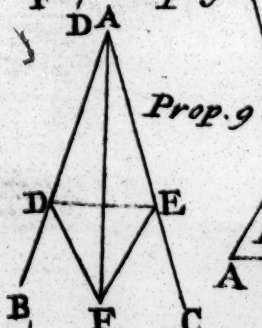
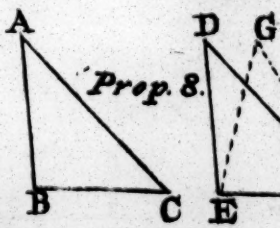
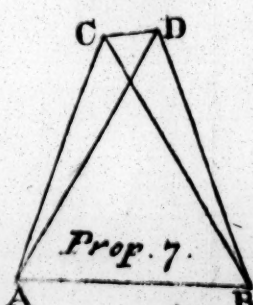
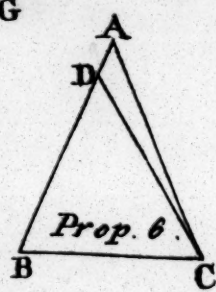
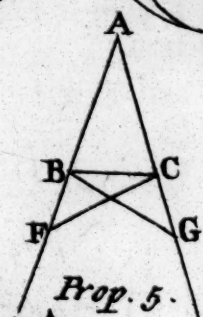
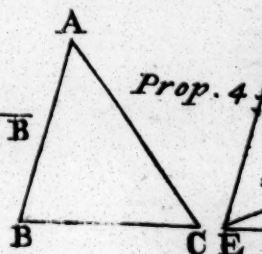
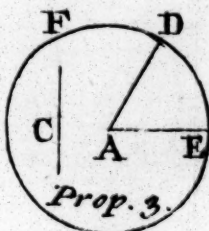
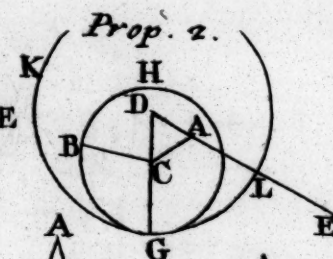
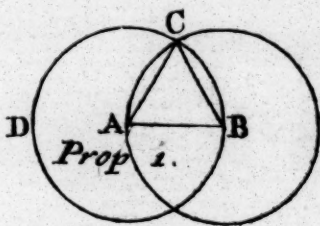
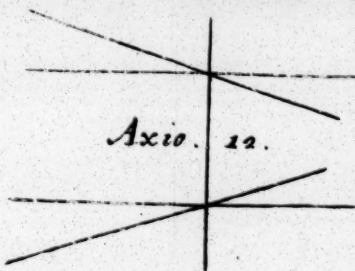
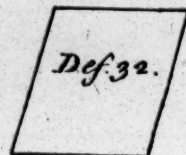
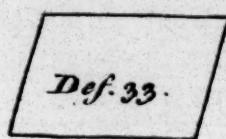
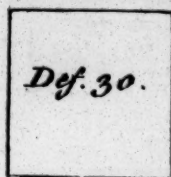
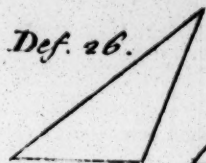
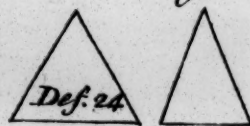


Def. 10.

Def. 11.



Def. 25.



PROPOSITION XIX.

THEOREM.

The greater Angle of every Triangle subtends the greater Side.

LET ABC be a Triangle, having the Angle ABC greater than the Angle BCA. I say, the Side AC is greater than the Side AB.

For, if it be not greater, AC is either equal to AB, or less than it. It is not equal to it, because then the Angle ABC would be equal * to the Angle ACB; * 5 of this, but it is not: Therefore AC is not equal to AB. Neither will it be less; for then the Angle ABC would be † less than the Angle ACB; but it is not. There-† 13 of this. fore AC is not less than AB. But likewise it has been proved not to be equal to it: Wherefore AC is greater than AB. Therefore *the greater Angle of every Triangle subtends the greater Side*; which was to be demonstrated.

PROPOSITION XX.

THEOREM.

Two Sides of any Triangle, howsoever taken, are together greater than the third Side.

LET ABC be a Triangle: I say, two Sides thereof, howsoever taken, are together greater than the third Side; viz. the Sides BA, AC, are greater than the Side BC; and the Sides AB, BC, greater than the Side AC; and the Sides BC, CA, greater than the Side AB.

For produce BA to the Point D, so that AD be * equal to AC, and join DC. * 3 of this.

Then, because DA is equal to AC, the Angle ADC shall be equal † to the Angle ACD. But the Angle † 5 of this. BCD is greater than the Angle ACD. Wherefore the Angle BCD is greater than the Angle ADC; and because DCB is a Triangle, having the Angle BCD greater than the Angle BDC, and the greater Angle

* 19 of this. Angle subtends * the greater Side ; the Side DB will be greater than the Side BC . But DB is equal to BA and AC together. Wherefore the Sides BA , AC , together, are greater than the Side BC . In the same manner we demonstrate, that the Sides AB , BC , together, are greater than the Side CA ; and the Sides BC , CA , together, are greater than the Side AB . Therefore *two Sides of any Triangle, howsoever taken, are together greater than the third Side ; which was to be demonstrated.*

PROPOSITION XXI.

THEOREM.

If two Right Lines be drawn from the extreme Points of one Side of a Triangle to any Point within the same, these two Lines shall be less than the other two Sides of the Triangle, but contain a greater Angle.

FOR let two Right Lines BD , DC , be drawn from the Extremes B , C , of the Side BC of the Triangle ABC , to the Point D within the same. I say, BD , DC , are less than BA , AC , the other two Sides of the Triangle, but contain an Angle ADC greater than the Angle BAC .

For produce BD to E .

Then, because two Sides of every Triangle together
* 20 of this. are * greater than the third, BA , AE , the two Sides of the Triangle ABE , are greater than the Side BE . Now, add EC , which is common, and the Sides
† Ax. 4. BA , AC , will be † greater than BE , EC .

Again, because CE , ED , the two Sides of the Triangle CED , are greater than the Side CD , add DB , which is common, and the Sides CE , EB , will be greater than CD , DB . But it has been proved, that BA , AC , are greater than BE , EC : Wherefore BA , AC , are much greater than BD , DC . Again, because
‡ 26 of this. the outward Angle of every Triangle is ‡ greater than the inward and opposite one ; ADC , the outward Angle of the Triangle CDE , shall be greater than the Angle CED . For the same Reason, CEB , the outward Angle of the Triangle ABE , is likewise greater
than

than the Angle BAC; but the Angle BDC has been proved to be greater than the Angle CEB. Wherefore the Angle BDC shall be much greater than the Angle BAC. And so, *if two Right Lines be drawn from the extreme Points of one Side of a Triangle to any Point within the same, these two Lines shall be less than the other two Sides of the Triangle, but contain a greater Angle*; which was to be demonstrated.

PROPOSITION XXII.

PROBLEM.

To describe a Triangle of three Right Lines, which are equal to three others given: But it is requisite, that any two of the Right Lines taken together be greater than the third; because two Sides of a Triangle, howsoever taken, are together greater than the third Side.

LET A, B, C, be three Right Lines given, two of which, any ways taken, are greater than the third, viz. A and B together greater than C; A and C greater than B; and B and C greater than A. Now it is required to make a Triangle of three Right Lines equal to A, B, C: Let there be one Right Line DE, terminated at D, but infinite towards E; and take * DF * 3 of this. equal to A, FG equal to B, and GH equal to C; and about the Centre F, with the Distance FD, describe a Circle DKL †; and about the Centre G, with † 3 Poss. the Distance GH, describe another Circle K L H, and join KF, KG. I say, the Triangle KFG is made of three Right Lines, equal to A, B, C; for, because the Point F is the Centre of the Circle DK, FK shall be equal to FD: But FD is equal to A; therefore FK is also equal to A. Again, because the Point G is the Centre of the Circle LKH, GK will be † † Def. 15. equal to GH: But GH is equal to C; therefore shall GK be also equal to C: But FG is likewise equal to B; and consequently the three Right Lines KF, FG, KG, are equal to the three Right Lines A, B, C; wherefore the Triangle KFG is made of three Right Lines KF, FG, GK, equal to the three given Lines A, B, C; which was to be done.

Euclid's ELEMENTS. Book I.
PROPOSITION XXIII.

PROBLEM.

With a given Right Line, and at a given Point in it, to make a Right-lin'd Angle equal to a Right-lin'd Angle given.

LET the given Right Line be AB , and the Point given therein A , and the given Right-lin'd Angle DCE . It is required to make a Right-lin'd Angle at the given Point A , with the given Right Line AB , equal to the given Right-lin'd Angle DCE .

Assume the Points D and E at Pleasure in the Lines CD , CE , and draw DE ; then, of three Right Lines equal to CD , DE , EC , make * a Triangle AFG , so that AF be equal to CD , AG to CE , and FG to DE .

Then, because the two Sides DC , CE , are equal to the two Sides FA , AG , each to each, and the Base DE equal to the Base FG ; the Angle DCE shall be † equal to the Angle FAG . Therefore the Right-lined Angle FAG is made at the given Point A , in the given Line AB , equal to the given Right-lined Angle DCE ; which was to be done.

PROPOSITION XXIV.

THEOREM.

If two Triangles have two Sides of the one, equal to two Sides of the other, each to each, and the Angle of the one, contained under the equal Right Lines, greater than the correspondent Angle of the other; then the Base of the one will be greater than the Base of the other.

LET there be two Triangles ABC , DEF , having two Sides AB , AC , equal to the two Sides DE , DF , each to each, viz. the Side AB equal to the Side DE , and the Side AC equal to DF ; and let the Angle BAC be greater than the Angle EDF . I say, the Base BC is greater than the Base EF .

Fo

For, because the Angle BAC is greater than the Angle EDF ; make an $\ast$ Angle EDG at the Point $\ast$ *23 of this.* D in the Right Line DE , equal to the Angle BAC ; and make $\dagger$ DG equal to either AC or DF , and $\dagger$ *3 of this.* join EF , FG .

Now, because AB is equal to DE , and AC to DG , the two Sides BA , AC , are each equal to the two Sides ED , DG , and the Angle BAC equal to the Angle EDG : Therefore the Base BC is equal $\dagger$ to $\dagger$ *4 of this.* the Base EG . Again, because DG is equal to DF , the Angle DFG is $\dagger$ equal to the Angle DGF ; and $\dagger$ *5 of this.* so the Angle DFG is greater than the Angle EGF : And consequently the Angle EFG is much greater than the Angle EGF . And because EFG is a Triangle, having the Angle EFG greater than the Angle EGF ; and the greatest Side subtends $\parallel$ the greatest $\parallel$ *19 of this.* Angle, the Side EG shall be greater than the Side EF . But the Side EG is equal to the Side BC : Whence BC is likewise greater than EF . Therefore, *if two Triangles have two Sides of the one, equal to two Sides of the other, each to each, and the Angle of the one, contained under the equal Right Lines, greater than the correspondent Angle of the other; then the Base of the one will be greater than the Base of the other; which was to be demonstrated.*

PROPOSITION XXV.

THEOREM.

If two Triangles have two Sides of the one equal to two Sides of the other, each to each, and the Base of the one greater than the Base of the other; they shall also have the Angles contained under the equal Sides, the one greater than the other.

LET there be two Triangles ABC , DEF , having two Sides AB , AC , each equal to two Sides DE , DF , viz. the Side AB equal to the Side DE , and the Side AC to the Side DF ; but the Base BC greater than the Base EF . I say, the Angle BAC is also greater than the Angle EDF .

For if it be not greater, it will be either equal or less. But the Angle BAC is not equal to the Angle

C 3

EDF ;

* 4 of this. EDF; for if it was, the Base BC would be * equal to the Base EF; but it is not: Therefore the Angle BAC is not equal to the Angle EDF, neither will it be lesser; for if it should, the Base BC would be † less than the Base EF; but it is not. Therefore the Angle BAC is not less than the Angle EDF; but it has likewise been proved not to be equal to it. Wherefore the Angle BAC is necessarily greater than the Angle EDF. *If, therefore, two Triangles have two Sides of the one equal to two Sides of the other, each to each, and the Base of the one greater than the Base of the other; they shall also have the Angles, contained under the equal Sides, the one greater than the other; which was to be demonstrated.*

PROPOSITION XXVI.

THEOREM.

If two Triangles have two Angles of the one equal to two Angles of the other, each to each, and one Side of the one equal to one Side of the other, either the Side lying between the equal Angles, or which subtends one of the equal Angles; the remaining Sides of the one Triangle shall be also equal to the remaining Sides of the other, each to his correspondent Side, and the remaining Angle of the one, equal to the remaining Angle of the other.

LET there be two Triangles ABC, DEF, having two Angles ABC, BCA, of the one, equal to two Angles DEF, EFD, of the other, each to each, that is, the Angle ABC equal to the Angle DEF, and the Angle BCA equal to the Angle EFD. And let one Side of the one be equal to one Side of the other, which first let be the Side lying between the equal Angles, viz. the Side BC equal to the Side EF. I say, the remaining Sides of the one Triangle will be equal to the remaining Sides of the other, each to each, that is, the Side AB equal to the Side DE, and the Side AC equal to the Side DF, and the remaining Angle BAC equal to the remaining Angle EDF.

For if the Side AB be not equal to the Side DE , one of them will be the greater, which let be AB , make GB equal to DE , and join GC .

Then, because BG is equal to DE , and BC to EF , the two Sides GB, BC , are equal to the two Sides DE, EF , each to each; and the Angle GBC equal to the Angle DEF . The Base GC is * equal to the Base DF , and the Triangle GBC to the Triangle DEF , and the remaining Angles equal to the remaining Angles, each to each, which subtend the equal Sides. Therefore the Angle GCB is equal to the Angle DFE . But the Angle DFE , by the Hypothesis, is equal to the Angle BCA ; and so the Angle BCG is likewise equal to the Angle BCA , the less to the greater, which cannot be. Therefore AB is not unequal to DE , and consequently is equal to it. And so the two Sides AB, BC , are each equal to the two Sides DE, EF , and the Angle ABC equal to the Angle DEF : And consequently the Base AC * is equal to the Base DF , and the remaining Angle BAC equal to the remaining Angle EDF .

Secondly, Let the Sides that are subtended by the equal Angles be equal, as AB equal to DE . I say, the remaining Sides of the one Triangle are equal to the remaining Sides of the other, *viz.* AC to DF , and BC to EF ; and also the remaining Angle BAC , to the remaining Angle EDF .

For if BC be unequal to EF , one of them is the greater, which let be BC , if possible, and make BH equal to EF , and join AH .

Now, because BH is equal to EF , and AB to DE , the two Sides AB, BH , are equal to the two Sides DE, EF , each to each, and they contain equal Angles; Therefore the Base AH is * equal to the Base DF ; and the Triangle ABH shall be equal to the Triangle DEF , and the remaining Angles equal to the remaining Angles, each to each, which subtend the equal Sides: And so the Angle BHA is equal to the Angle EFD . But EFD is † equal to the Angle BCA ; and consequently the Angle BHA is equal to the Angle BCA : Therefore the outward Angle BHA of the Triangle AHC , is equal to the inward and opposite Angle BCA ; which is † impossible: † 16 of this. Whence BC is not unequal to EF ; therefore it is

equal to it. But AB is also equal to DE . Wherefore the two Sides AB , BC , are equal to the two Sides DE , EF , each to each; and they contain equal Angles. And so the Base AC is equal to the Base DF , the Triangle BAC to the Triangle DEF , and the remaining Angle BAC equal to the remaining Angle EDF . *If, therefore, two Triangles have two Angles equal, each to each, and one Side of the one equal to one Side of the other, either the Side lying between the equal Angles, or which subtends one of the equal Angles; the remaining Sides of the one Triangle shall be also equal to the remaining Sides of the other, each to his correspondent Side, and the remaining Angle of the one equal to the remaining Angle of the other; which was to be demonstrated.*

PROPOSITION XXVII.

THEOREM.

If a Right Line, falling upon two Right Lines, makes the alternate Angles equal between themselves, the two Right Lines shall be parallel.

LET the Right Line EF , falling upon two Right Lines AB , CD , make the alternate Angles AEF , EFD , equal between themselves. I say, the Right Line AB is parallel to CD .

For if it be not parallel, AB and CD , produced towards B and D , or towards A and C , will meet: Now let them be produced towards B and D , and meet in the Point G .

* 16 of this.
† from the
Hyp.

Then the outward Angle AEF of the Triangle GEF is * greater than the inward and opposite Angle EFG , and also equal † to it; which is absurd. Therefore AB and CD , produced towards B and D , will not meet each other. By the same way of Reasoning, neither will they meet, being produced towards C and A . But Lines that meet each other on neither Side, are † parallel between themselves. Therefore AB is parallel to CD . Therefore, *if a Right Line, falling upon two Right Lines, makes the alternate Angles equal between themselves, the two Right Lines shall be parallel, which was to be demonstrated.*

‡ Def. 35.

PRO-

PROPOSITION XXVIII.

THEOREM.

If a Right Line falling upon two Right Lines, makes the outward Angle of the one Line equal to the inward and opposite Angle of the other on the same Side, or the inward Angles on the same Side together equal two Right Angles, the two Right Lines shall be parallel between themselves.

LET the Right Line EF, falling upon two Right Lines AB, CD, make the outward Angle EGB equal to the inward and opposite Angle GHD; or the inward Angles BGH, GHD, on the same Side together equal to two Right Angles. I say, the Right Line AB is parallel to the Right Line CD.

For, because the Angle EGB is * equal to the Angle ^{From the} GHD, and the Angle EGB † equal to the Angle ^{Hyp.} AGH, the Angle AGH shall be equal to the Angle ^{† 15 of this.} GHD; but these are Alternate Angles. Therefore AB is ‡ parallel to CD.

Again, because the Angles BGH, GHD, are equal ^{† 27 of this.} to two Right Angles, and AGH, BGH, are * equal ^{* 13 of this.} to two Right ones, the Angles AGH, BGH, will be equal to the Angles BGH, GHD; and if the common Angle BGH be taken from both, there will remain the Angle AGH equal to the Angle GHD; but these are alternate Angles. Therefore AB is parallel to CD. *If, therefore, a Right Line, falling upon two Right Lines, makes the outward Angle of the one Line equal to the inward and opposite Angle of the other on the same Side, or the inward Angles on the same Side together equal to two Right Angles, the two Right Lines shall be parallel between themselves; which was to be demonstrated.*

PROPOSITION XXIX.

THEOREM.

If a Right Line falls upon two Parallels, it will make the alternate Angles equal between themselves; the outward Angle equal to the inward and opposite Angle, on the same Side; and the inward Angles on the same Side together equal to two Right Angles.

LET the Right Line EF fall upon the parallel Right Lines AB, CD. I say, the alternate Angles AGH, GHD, are equal between themselves; the outward Angle EGB is equal to the inward one GHD, on the same Side; and the two inward ones, BGH, GHD, on the same Side, are together equal to two Right Angles.

For if AGH be unequal to GHD, one of them will be the greater. Let this be AGH; then, because the Angle AGH is greater than the Angle GHD, add the common Angle BGH to both: And so the Angles AGH, BGH, together, are greater than the Angles BGH, GHD, together. But the Angles AGH, BGH, are equal to two Right ones*. Therefore BGH, GHD, are less than two Right Angles. † Ax. 12. And so the Lines AB, CD, infinitely produced †, will meet each other; but because they are parallel, they will not meet. Therefore the Angle AGH is not unequal to the Angle GHD. Wherefore it is necessarily equal to it.

‡ 15 of this. But the Angle AGH is ‡ equal to the Angle EGB: Therefore EGB is also equal to GHD.

Now add the common Angle BGH, and then EGB, BGH, together, are equal to BGH, GHD, together; but EGB, and BGH, are equal to two Right Angles. Therefore also BGH, and GHD, shall be equal to two Right Angles. Wherefore, if a Right Line falls upon two Parallels, it will make the alternate Angles equal between themselves; the outward Angle equal to the inward and opposite Angle, on the same Side, and the inward Angles on the same Side together equal to two Right Angles; which was to be demonstrated.

PRO-

PROPOSITION XXX.

THEOREM.

Right Lines parallel to one and the same Right Line, are also parallel between themselves.

LET AB and CD be Right Lines, each of which is parallel to the Right Line EF. I say, AB is also parallel to CD. For let the Right Line GK fall upon them.

Then, because the Right Line GK falls upon the parallel Right Lines AB, EF, the Angle AGH is * * ^{29 of this.} equal to the Angle GHF; and because the Right Line GK falls upon the parallel Right Lines EF, CD, the Angle GHF is equal to the Angle GKD*. But it has been proved, that the Angle AGK is also equal to the Angle GHF. Therefore AGK is equal to GKD, and they are alternate Angles; whence AB is parallel to CD†. And so *Right Lines parallel to one and the same Right Line, are parallel between themselves*; which was to be demonstrated. ^{† 27 of this.}

PROPOSITION XXXI.

PROBLEM.

To draw a Right Line thro' a given Point parallel to a given Right Line.

LET A be a Point given, and BC a Right Line given. It is required to draw a Right Line thro' the Point A, parallel to the Right Line BC.

Assume any Point D in BC, and join AD; then make * an Angle DAE, at the Point A, with the ^{* 23 of this.} Line DA, equal to the Angle ADC, and produce EA strait forwards to F.

Then, because the Right Line AD, falling on two Right Lines BC, EF, makes the alternate Angles EAD, ADC, equal between themselves, EF shall be † parallel to BC. Therefore the Right Line EAF ^{† 27 of this.} is drawn thro' the given Point A, parallel to the given Right Line BC; which was to be done.

Coroll.

Coroll. Hence it appears, that if one Angle of any Triangle be equal to the other two, that is a Right one ; because that the Angle adjacent to this Right one, is equal to the other two. But when adjacent Angles are equal, they are necessarily Right ones.

P R O P O S I T I O N X X X I I .

T H E O R E M .

If one Side of any Triangle be produced, the outward Angle is equal to both the inward and opposite Angles ; and the three inward Angles of a Triangle are equal to two Right Angles.

LET ABC be a Triangle, one of whose Sides BC is produced to D . I say, the outward Angle ACD is equal to the two inward and opposite Angles CAB, ABC ; and the three inward Angles of the Triangle, *viz.* ABC, BCA, CAB , are equal to two Right Angles.

* 31 of this. For let CE be drawn * thro' the Point C , parallel to the Right Line AB . Then, because AB is parallel to CE , and AC falls upon them, the alternate Angles BAC, ACE , are † equal between themselves. Again, † 29 of this. because AB is parallel to CE , and the Right Line BD falls upon them, the outward Angle ECD is † equal to the inward and opposite one ABC ; but it has been proved, that the Angle ACE is equal to the Angle BAC . Wherefore the whole outward Angle ACD is equal to both the inward and opposite Angles BAC, ABC . And if the Angle ACB , which is common, be added, the two Angles ACD, ACB , are equal to the three Angles ABC, BAC, ACB ; but the Angles ACD, ACB , are ‡ equal to two Right Angles. Therefore also shall the ‡ 13 of this. Angles ACB, CBA, CAB , be equal to two Right Angles. Wherefore, *if one Side of any Triangle be produced, the outward Angle is equal to both the inward and opposite Angles, and the three inward Angles of a Triangle are equal to two Right Angles ; which was to be demonstrated.*

Coroll.

Coroll. 1. All the three Angles of any one Triangle taken together, are equal to all the three Angles of any other Triangle taken together.

Coroll. 2. If two Angles of any one Triangle, either separately or taken together, be equal to two Angles of any other Triangle; then the remaining Angle of the one Triangle will be equal to the remaining Angle of the other.

Coroll. 3. If one Angle of a Triangle be a Right Angle, the other two Angles together make one Right Angle.

Coroll. 4. If the Angle included between the equal Legs of an Isosceles Triangle be a Right one, each of the other Angles at the Base will be half Right Angles.

Coroll. 5. Any Angle in an Equilateral Triangle is equal to one Third of two Right Angles, or two Thirds of one Right Angle.

THEOREM I.

All the inward Angles of any Right-lin'd Figure whatsoever, make twice as many Right Angles, abating four, as the Figure has Sides.

FOR any Right-lin'd Figure may be resolved into as many Triangles, abating two, as it hath Sides. For Example, if a Figure has four Sides, it may be resolved into two Triangles: If a Figure has five Sides, it may be resolved into three Triangles; if six, into four; and so on. Wherefore (by Prop. XXXII.) the Angles of all these Triangles are equal to twice as many Right Angles as there are Triangles: But the Angles of all the Triangles are equal to the inward Angles of the Figure. Therefore all the inward Angles of the Figure are equal to twice as many Right Angles as there are Triangles, that is, twice as many Right Angles, taking away four, as the Figure has Sides.
W. W. D.

THEOREM II.

All the outward Angles of any Right-lin'd Figure together, make four Right Angles.

FOR the outward Angles, together with the inward ones, make twice as many Right Angles as the Figure has Sides; but from the last Theorem, all the inward Angles together make twice as many Right Angles, abating four, as the Figure has Sides. Wherefore the outward Angles are all together equal to four Right Angles. W. W. D.

PROPOSITION XXXII.

THEOREM.

Two Right Lines, which join two equal and parallel Right Lines, towards the same Parts, are also equal and parallel.

LET the parallel and equal Right Lines AB, CD, be joined towards the same Parts, by the Right Lines AC, BD. I say, AC, BD, are equal and parallel.
For draw BC.

Then, because AB is parallel to CD, and BC falls upon them, the alternate Angles ABC, BCD, are
* 29 of this. * equal. Again, because AB is equal to CD, and BC is common; the two Sides AB, BC, are each equal to the two Sides BC, CD; but the Angle ABC is also equal to the Angle BCD; therefore the
† 4 of this. Base AC is † equal to the Base BD: And the Triangle ABC, equal to the Triangle BCD; and the remaining Angles equal to the remaining Angles, each to each, which subtend the equal Sides. Wherefore the Angle ACB is equal to the Angle CBD. And because the Right Line BC, falling upon two Right Lines AC, BD, make ‡ the alternate Angles ACB, CBD, equal
‡ 27 of this. between themselves; AC is ‡ parallel to BD. But it has been proved also to be equal to it. Therefore two Right Lines, which join two unequal and parallel Right Lines, towards the same Parts, are also equal and parallel; which was to be demonstrated.

Defin-

Defin. *A Parallelogram is a Quadrilateral Figure, each of whose opposite Sides are parallel.*

PROPOSITION XXXIV.

THEOREM.

The opposite Sides and opposite Angles of any Parallelogram are equal; and the Diameter divides the same into two equal Parts.

LET $ABDC$ be a Parallelogram, whose Diameter is BC . I say, the opposite Sides and opposite Angles are equal between themselves, and the Diameter BC bisects the Parallelogram.

For, because AB is parallel to CD , and the Right Line BC falls on them, the alternate Angles ABC , BCD , are * equal between themselves; again, because AC is parallel to BD , and BC falls upon them, the alternate Angles ACB and CBD are equal to one another. Wherefore ABC , CBD , are two Triangles, having two Angles ABC , BCA , of the one, equal to two Angles BCD , CBD , of the other, each to each; and likewise one Side of the one equal to one Side of the other, viz. the Side BC between the equal Angles, which is common. Therefore the remaining Sides shall be † equal to the remaining Sides, each to each, and the remaining Angle to the remaining Angle. And so the Side AB is equal to the Side CD , the Side AC to BD , and the Angle BAC to the Angle BDC . And because the Angle ABC is equal to the Angle BCD , and the Angle CBD to the Angle ACB ; therefore the whole Angle ABD is equal to the whole Angle ACD : But it has been proved, that the Angle BAC is also equal to the Angle BDC .

Wherefore the opposite Sides and Angles of any Parallelogram are equal between themselves.

I say, moreover, that the Diameter bisects it. For because AB is equal to CD , and BC is common, the two Sides AB , BC , are each equal to the two Sides DC , CB ; and the Angle ABC is also equal to the Angle BCD . Therefore the Base AC is ‡ equal to the Base DB ; and the Triangle ABC is † 4 of this. equal

equal to the Triangle $\triangle BCD$. Wherefore the Diameter BC bisects the Parallelogram $ACDB$; which was to be demonstrated.

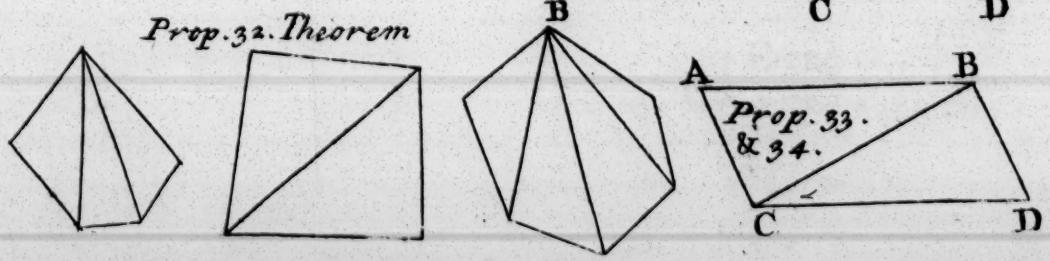
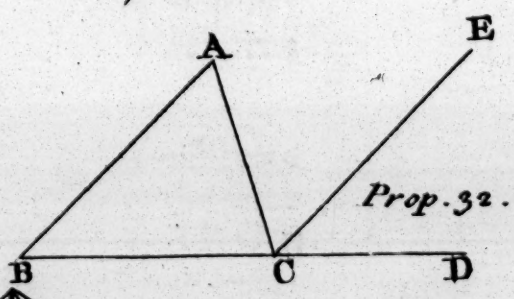
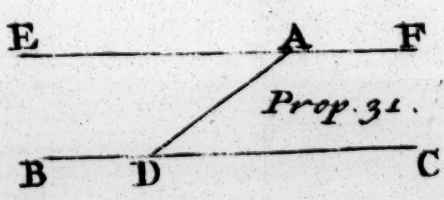
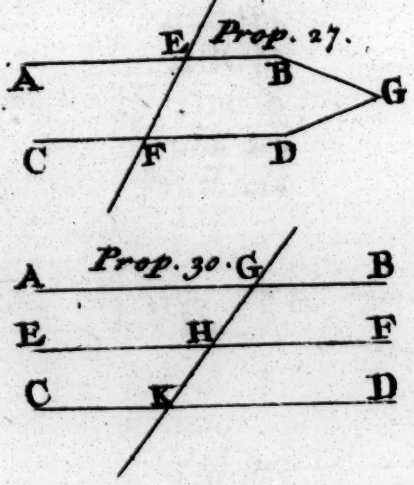
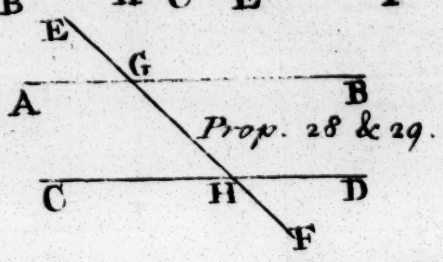
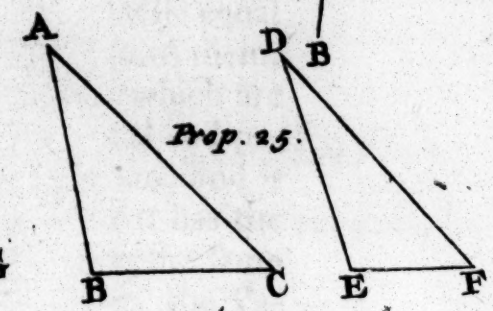
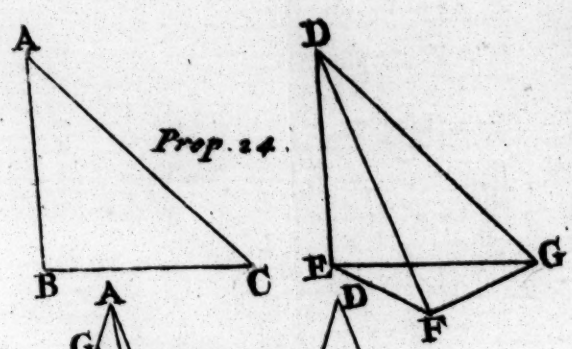
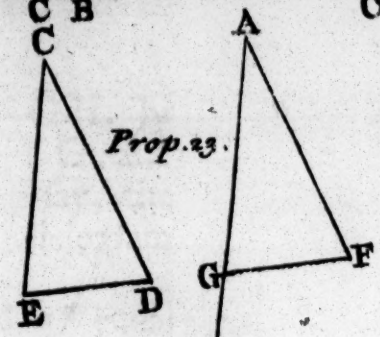
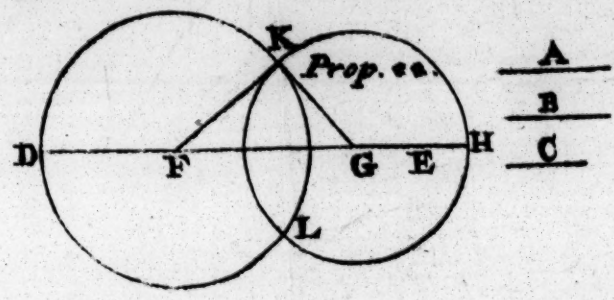
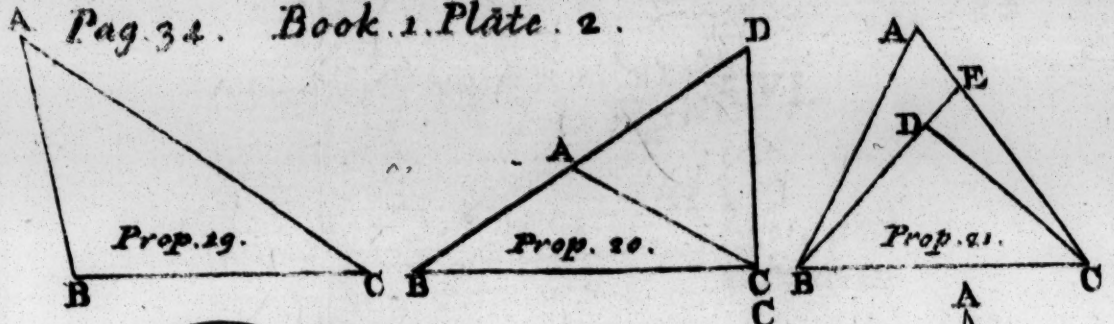
PROPOSITION XXXV.

THEOREM.

Parallelograms constituted upon the same Base, and between the same Parallels, are equal between themselves.

LET $ABCD$, $EBCF$, be Parallelograms constituted upon the same Base BC , and between the same Parallels AF and BC . I say, the Parallelogram $ABCD$ is equal to the Parallelogram $EBCF$.

For because $ABCD$ is a Parallelogram, AD is
 * 34 of this. * equal to BC ; and for the same Reason EF is equal
 † Ax. 1. to BC ; wherefore AD shall be † equal to EF ; but
 † Ax. 2. DE is common. Therefore the Whole AE is † equal
 to the Whole DF . But AB is equal to DC ; wherefore
 EA , AB , the two Sides of the Triangle EAB ,
 are equal to the two Sides FD , DC , each to each;
 * 29 of this. and the Angle FDC * equal to the Angle EAB , the
 outward one to the inward one. Therefore the Base
 † 4 of this. EB is † equal to the Base CF , and the Triangle EAB
 to the Triangle FDC . If the common Triangle
 † Ax. 3. DGE be taken from both, there will remain † the
 Trapezium $ABGD$, equal to the Trapezium $FCGE$;
 and if the Triangle GBC , which is common, be
 added, the Parallelogram $ABCD$ will be equal to
 the Parallelogram $EBCF$. Therefore *Parallelo-*
grams constituted upon the same Base, and between
the same Parallels, are equal between themselves;
 which was to be demonstrated.





PROPOSITION XXXVI.

THEOREM.

Parallelograms constituted upon equal Bases, and between the same Parallels, are equal between themselves.

LET the Parallelograms $ABCD$, $EFGH$, be constituted upon the equal Bases BC , FG , and between the same Parallels AH , BG . I say, the Parallelogram $ABCD$ is equal to the Parallelogram $EFGH$.

For join BE , CH . Then, because BC is * equal * *Hyp*, to FG , and FG to EH ; BC will be likewise equal to EH ; and they are parallel, and BE , CH , joins them. But two Right Lines joining Right Lines which are equal and parallel the same way, are † equal and parallel: Wherefore $EBCH$ is a Parallelogram, and is † equal to the Parallelogram $ABCD$; for it has the same Base BC , and is constituted between the same Parallels BC , AD . For the same Reason, the Parallelogram $EFGH$ is equal to the same Parallelogram $EBCH$. Therefore the Parallelogram $ABCD$ shall be equal to the Parallelogram $EFGH$. And so *Parallelograms constituted upon equal Bases, and between the same Parallels, are equal between themselves*; which was to be demonstrated.

PROPOSITION XXXVII.

THEOREM.

Triangles constituted upon the same Base, and between the same Parallels, are equal between themselves.

LET the Triangles ABC , DBC , be constituted upon the same Base BC , and between the same Parallels AD , BC . I say, the Triangle ABC is equal to the Triangle DBC .

For produce AD both ways to the Points E and F ; and through B draw * BE parallel to CA ; and * CF parallel to BD .

D

Where-

Wherefore both $EBCA$, $DBCF$, are Parallelograms; and the Parallelogram $EBCA$ is * equal to the Parallelogram $DBCF$; for they stand upon the same Base BC , and between the same Parallels BC , EF . But the Triangle ABC is † one half of the Parallelogram $EBCA$, because the Diameter AB bisects it; and the Triangle DBC is one half of the Parallelogram $DBCF$; for the Diameter DC bisects it. But Things that are the Halves of equal Things, ‡ *Axiom 7.* are ‡ equal between themselves. Therefore the Triangle ABC is equal to the Triangle DBC . Wherefore, *Triangles constituted upon the same Base, and between the same Parallels, are equal between themselves*; which was to be demonstrated.

PROPOSITION XXXVIII.

THEOREM.

Triangles constituted upon equal Bases, and between the same Parallels, are equal between themselves.

LET the Triangles ABC , DCE , be constituted upon the equal Bases BC , CE , and between the same Parallels BE , AD . I say, the Triangle ABC is equal to the Triangle DCE .

For, produce AD both ways to the Points G , H ; thro' B draw * BG parallel to CA ; and thro' E , EH , parallel to DC .

Wherefore both $GBCA$, $DCEH$, are Parallelograms; and the Parallelogram $GBCA$ is † equal to the Parallelogram $DCEH$. For they stand upon equal Bases, BC , CE , and between the same Parallels BE , GH . But the Triangle ABC is ‡ one half of the Parallelogram $GBCA$; for the Diameter AB bisects it; and the Triangle DCE ‡ is one half of the Parallelogram $DCEH$; for the Diameter DE bisects it. But Things that are the Halves of equal Things, are * equal between themselves. Therefore the Triangle ABC is equal to the Triangle DCE . Wherefore *Triangles constituted upon equal Bases, and between the same Parallels, are equal between themselves*; which was to be demonstrated.

P R O -

PROPOSITION XXXIX.

THEOREM.

Equal Triangles constituted upon the same Base, on the same Side, are in the same Parallels.

LET ABC, DBC , be equal Triangles, constituted upon the same Base BC , on the same Side. I say, they are between the same Parallels. For, let AD be drawn. I say, AD is parallel to BC .

For, if it be not parallel, draw * the Right Line AE * ^{31 of this.} thro' the Point A , parallel to BC , and draw EC .

Then the Triangle ABC † is equal to the Triangle † ^{37 of this.} EBC ; for it is upon the same Base BC , and between the same Parallels BC, AE . But the Triangle ABC is † equal to the Triangle DBC . Therefore the † ^{From Hyp.} Triangle DBC is also equal to the Triangle EBC , a less to a greater, which is impossible. Wherefore AE is not parallel to BC : And by the same way of reasoning we prove, that no other Line but AD is parallel to BC . Therefore AD is parallel to BC . Wherefore *equal Triangles constituted upon the same Base, on the same Side, are in the same Parallels;* which was to be demonstrated.

PROPOSITION XL.

THEOREM.

Equal Triangles constituted upon equal Bases, on the same Side, are between the same Parallels.

LET ABC, CDE , be equal Triangles, constituted upon equal Bases BC, CE . I say, they are between the same Parallels. For, let AD be drawn. I say, AD is parallel to BE .

For, if it be not, let AF be drawn * thro' A , parallel to BE , and draw FE . * ^{31 of this.}

Then the Triangle ABC is † equal to the Triangle † ^{38 of this.} FCE ; for they are constituted upon equal Bases, and between the same Parallels BE, AF . But the Triangle ABC is equal to the Triangle DCE . Therefore

D 2

fore

fore the Triangle DCE shall be equal to the Triangle FCE, the greater to the less, which is impossible. Wherefore AF is not parallel to BE. And in this manner we demonstrate, that no Right Line can be parallel to BE, but AD. Therefore AD is parallel to BE. And so *equal Triangles constituted upon equal Bases, on the same Side, are between the same Parallels*; which was to be demonstrated.

PROPOSITION XLI.

PROBLEM.

If a Parallelogram and a Triangle have the same Base, and are between the same Parallels, the Parallelogram will be double to the Triangle.

LET the Parallelogram ABCD, and the Triangle EBC, have the same Base, and be between the same Parallels, BC, AE. I say, the Parallelogram ABCD is double the Triangle EBC.

For join AC.

* 37 of this. Now the Triangle ABC is * equal to the Triangle EBC; for they are both constituted upon the same Base BC, and between the same Parallels BC, AE.

† 34 of this. But the Parallelogram ABCD is † double the Triangle ABC, since the Diameter AC bisects it. Wherefore likewise it shall be double to the Triangle EBC. *If, therefore, a Parallelogram and Triangle have both the same Base, and are between the same Parallels, the Parallelogram will be double to the Triangle*; which was to be demonstrated.

PROPOSITION XLII.

THEOREM.

To constitute a Parallelogram equal to a given Triangle, in an Angle equal to a given Right-lined Angle.

LET the given Triangle be ABC, and the Right-lined Angle given D. It is required to constitute a Parallelogram equal to the given Triangle ABC, in a Right-lined Angle equal to D. Bisect

Bisect * BC in E, join AE, and at the Point E, in * 10 of this.
the Right Line EC, constitute † an Angle CEF † 23 of this.
equal to D. Also draw ‡ AG thro' A, parallel to EC, ‡ 31 of this.
and thro' C the Right Line CG parallel to FE.

Now FECG is a Parallelogram: And because
BE is equal to EC, the Triangle ABE shall be * equal * 38 of this,
to the Triangle AEC; for they stand upon equal
Bases BE, EC, and are between the same Parallels
BC, AG. Wherefore the Triangle ABC is double
to the Triangle AEC. But the Parallelogram FECG
is also double to the Triangle AEC: for it has the
same Base, and is between the same Parallels. There-
fore the Parallelogram FECG, is equal to the Tri-
angle ABC, and has the Angle CEF equal to the
Angle D. Wherefore the Parallelogram FECG is
constituted equal to the given Triangle ABC, in an
Angle CEF equal to a given Angle D; *which was
to be done.*

PROPOSITION XLIII.

THEOREM.

*In every Parallelogram the Complements of the
Parallelograms that stand about the Diameter,
are equal between themselves.*

LET ABCD be a Parallelogram, whose Diame-
ter is DB; and let FH, EG, be Parallelograms
standing about the Diameter BD. Now AK, KC,
are called the Complements of them: I say, the Com-
plement AK is equal to the Complement KC.

For since ABCD is a Parallelogram, and BD is
the Diameter thereof, the Triangle ABD * is equal * 34 of this,
to the Triangle BDC. Again, because HKFD is
a Parallelogram, whose Diameter is DK, the Trian-
gle HDK shall * be equal to the Triangle DFK; and
for the same Reason the Triangle KBG is equal to
the Triangle KEB. But since the Triangle BEK is
equal to the Triangle BGK, and the Triangle HDK
to DFK; the Triangle BEK, together with the
Triangle HDK, is equal to the Triangle BGK, to-
gether with the Triangle DFK. But the whole Tri-
angle ABD is likewise equal to the whole Triangle
BDC,

BDC. Wherefore the Complement remaining, AK, will be equal to the remaining Complement KC. Therefore *in every Parallelogram the Complements of the Parallelograms, that stand about the Diameter, are equal between themselves; which was to be done.*

PROPOSITION XLIV.

PROBLEM.

To apply a Parallelogram to a given Right Line, equal to a given Triangle, in a given Right-lined Angle.

LET the Right Line given be AB, the given Triangle C, and the given Right-lined Angle D. It is required, to the given Right Line AB, to apply a Parallelogram equal to the given Triangle C.

In an Angle equal to D, make the Parallelogram ** 42 of this.* BEFG equal to *** the Triangle C; in the Angle EBG, equal to D. Place BE in a strait Line with AB, and produce FG to H, and thro' A let AH be *† 31 of this.* drawn *†* parallel to either GB, or FE, and join HB.

Now, because the Right Line HF falls on the *‡ 29 of this.* parallels AH, EF, the Angles AHF, HFE, are *‡* equal to two Right Angles. And so BHF, HFE, are less than two Right Angles; but Right Lines making less than two Right Angles, with a third Line being ** Ax. 12.* infinitely produced, will meet *** each other. Wherefore HB, FE, produced, will meet each other; which ** 31 of this.* let be in K, thro' which *** draw KL parallel to EA, or FH, and produce AH, GB, to the Points L and M.

Therefore HLKF is a Parallelogram, whose Diameter is HK; and AG, ME, are Parallelograms about HK; whereof LB, BF, are the Complements. *† 43 of this.* Therefore LB is *†* equal to BF. But BF is also equal to the Triangle C. Wherefore likewise LB shall be equal to the Triangle C; and because the *‡ 15 of this.* Angle GBE is *‡* equal to the Angle ABM, and also equal to the Angle D, the Angle ABM shall be equal to the Angle D. Therefore to the given Right Line AB is applied a Parallelogram, equal to the given Triangle C, in the Angle ABM, equal to the given Angle D; *which was to be done.*

PROPOSITION XLV.

PROBLEM.

To make a Parallelogram equal to a given Right-lined Figure, in a given Right-lined Angle.

LET ABCD be the given Right-lined Figure, and E the Right-lined Angle given. It is required to make a Parallelogram equal to the Right-lined Figure ABCD in an Angle equal to E.

Let DB be joined, and make * the Parallelogram * 42 of this. FH equal to the Triangle ADB, in an Angle HKF, equal to the given Angle E.

Then to the Right Line GH apply † the Parallelogram GM, equal to the Triangle DBC, in an Angle GHM, equal to the Angle E. † 44 of this.

Now, because the Angle E is equal to HKF, or GHM, the Angle HKF shall be equal to GHM, and KHG to both; and the Angles HKF, KHG, are, together, equal to the Angles KHG, GHM. But HKF, KHG, are ‡, together, equal to two Right † 29 of this. Angles. Wherefore, likewise, the Angles KHG, GHM, shall be equal to two Right Angles: And so, at the given Point H in the Right Line GH, two Right Lines KH, HM, not drawn on the same Side, make the adjacent Angles, both together, equal to two Right Angles; and consequently KH, HM *, * 41 of this. make one strait Line. And because the Right Line HG falls upon the Parallels KM, FG, the alternate Angles MHG, HGF, are ‡ equal. And if HGL be added to both, the Angles MHG, HGL, together, are equal to the Angles HGF, HGL, together. But the Angles MHG, HGL, are * together equal * 29 of this. to two Right Angles. Wherefore, likewise, the Angles HGF, HGL, are together equal to two Right Angles; and so FG, GL, make one strait Line. And since KF is equal and parallel to HG, as likewise HG to ML, KF shall be † equal and parallel † 30 of this. to ML, and the Right Lines KM, FL, join them. Wherefore KM, FL, are ‡ equal and parallel. Therefore † 34 of this. KFLM is a Parallelogram. But since the Triangle ABD is equal to the Parallelogram HF, and

the Triangle DBC to the Parallelogram GM ; then the whole Right-lined Figure $ABCD$ will be equal to the whole Parallelogram $KFLM$. Therefore the Parallelogram $KFLM$ is made equal to the given Right-lined Figure $ABCD$, in an Angle FKM , equal to the given Angle E ; *which was to be done.*

Coroll. It is manifest, from what has been said, how to apply a Parallelogram to a given Right Line, equal to a given Right-lined Figure in a given Right-lined Angle.

PROPOSITION XLVI.

PROBLEM.

To describe a Square upon a given Right Line.

LET AB be the Right Line given, upon which it is required to describe a Square.

* 11 of this. Draw * AC at Right Angles to AB from the Point
† 3 of this. A given therein; make † AD equal to AB , and thro'
† 31 of this. the Point D draw † DE parallel to AB ; also thro'
 B draw BE parallel to AD .

* 34 of this. Then $ADEB$ is a Parallelogram; and so AB * is
equal to DE , and AD to BE . But BA is equal to
 AD . Therefore the four Sides BA , AD , DE , EB ,
are equal to each other.

And so the Parallelogram $ADEB$ is equilateral:
I say, it is likewise equiangular. For because the Right
Line AD falls upon the Parallels AB , DE , the An-
† 29 of this. gles BAD , ADE , are † equal to two Right Angles.
But BAD is a Right Angle: Wherefore ADE is
also a Right Angle; but the opposite Sides and oppo-
† 34 of this. site Angles of Parallelograms are † equal. Therefore
each of the opposite Angles ABE , BED , are Right
Angles; and consequently $ADBE$ is a Rectangle:
But it has been proved to be equilateral. Therefore
it is necessarily a Square, and is described upon the
Right Line AB ; *which was to be done.*

Coroll. Hence every Parallelogram, that has one Right Angle, is a Rectangle.

PROPOSITION XLVII.

THEOREM.

In any Right-angled Triangle, the Square described upon the Side subtending the Right Angle, is equal to both the Squares described upon the Sides containing the Right Angle.

LET ABC be a Right-angled Triangle, having the Right Angle BAC. I say, the Square described upon the Right Line BC, is equal to both the Squares described upon the Sides BA, AC.

For describe * upon BC the Square BDEC, and * *46 of this.* on BA, AC, the Squares GB, HC; and thro' the Point A draw AL parallel to BC, or CE; and let AD, FC, be joined.

Then, because the Angles BAC, BAG, † are Right † *Def. 30.* ones, two Right Lines AG, AC, at the given Point A, in the Right Line BA, being on contrary Sides thereof, make the adjacent Angles equal to two Right Angles. Therefore CA, AG, make † one strait † *14 of this.* Line; by the same Reason AB, AH, make one strait Line. And since the Angle DBC is equal to the Angle FBA, for each of them is a Right one, add ABC, which is common, and the whole Angle DBA is * equal to the whole Angle FBC. And * *Ax. 2.* since the two Sides AB, BD, are equal to the two Sides FB, BC, each to each, and the Angle DBA equal to the Angle FBC; the Base AD will be † † *4 of this.* equal to the Base FC, and the Triangle ABD equal to the Triangle FBC: But the Parallelogram BL is † double to the Triangle ABD; for they have the same Base DB, and are between the same Parallels BD, AL. The Square GB is † also double to the Triangle FBC; for they have the same Base FB, and are in the same Parallels FB, GC. But Things that are the Doubles of equal Things, are * equal to * *Ax. 6.* each other. Therefore the Parallelogram BL is equal to the Square GB. After the same manner, AE, BK, being joined, we prove, that the Parallelogram CL is equal to the Square HC. Therefore the whole Square DBEC is equal to the two Squares GB, HC. But the Square DBEC is described on the Right Line BC,

BC, and the Squares GB, HC, on BA, AC. Therefore the Square BE, described on the Side BC, is equal to the Squares described on the Sides BA, AC. Wherefore in any Right-angled Triangle, the Square described upon the Side subtending the Right Angle, is equal to both the Squares described upon the Sides containing the Right Angle.

PROPOSITION XLVIII.

THEOREM.

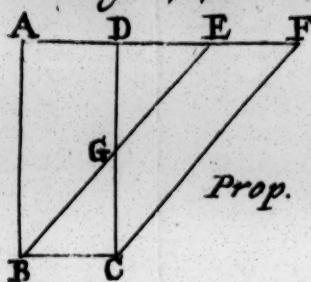
If a Square described upon one Side of a Triangle be equal to the Squares described upon the other two Sides of the said Triangle, then the Angle contained by these two other Sides is a Right Angle.

IF the Square described upon the Side BC of the Triangle ABC, be equal to the Squares described upon the other two Sides of the Triangle BA, AC: I say, the Angle BAC is a Right one.

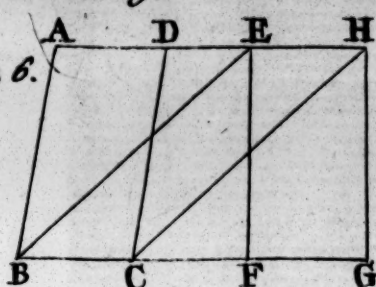
For, let there be drawn AD from the Point A, at Right Angles, to AC; likewise make AD equal to BA, and join DC.

Then, because DA is equal to AB, the Square described on DA will be equal to the Square described on AB. And adding the common Square described on AC, the Squares described on DA, AC, are equal to the Squares described on BA, AC. But the Square described on DC is * equal to the Squares described on DA, AC; for DAC is a Right Angle: But the Square on BC is put equal to the Squares on BA, AC. Therefore the Square described on DC is equal to the Square described on BC; and so the Side CD is equal to the Side CB. And because DA is equal to AB, and AC is common, the two Sides DA, AC, are equal to the two Sides BA, AC; and the Base DC is equal to the Base CB. Therefore the Angle DAC is † equal to the Angle BAC; but DAC is a Right Angle; and so BAC will be a Right Angle also. If, therefore, a Square described upon one Side of a Triangle be equal to the Squares described upon the other two Sides of the said Triangle, then the Angle contained by these two other Sides is a Right Angle; which was to be demonstrated.

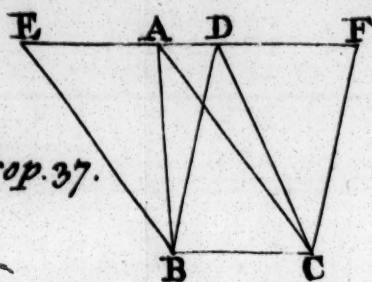
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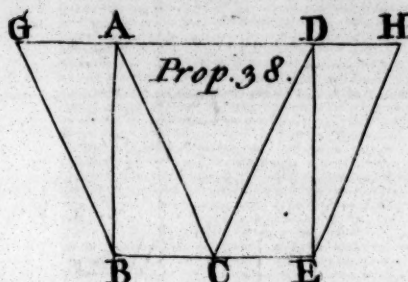
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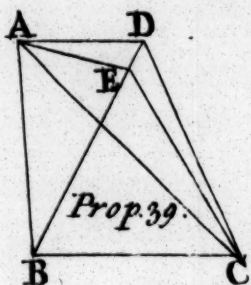
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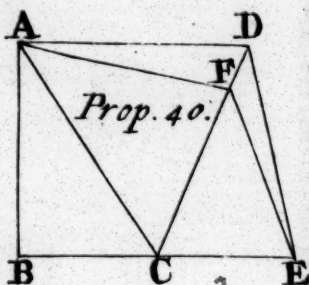
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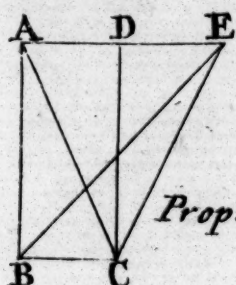
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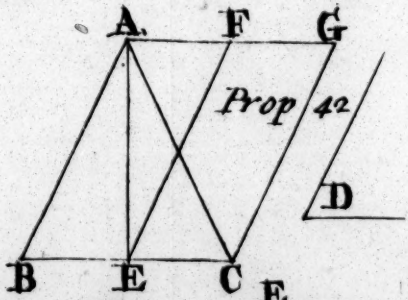
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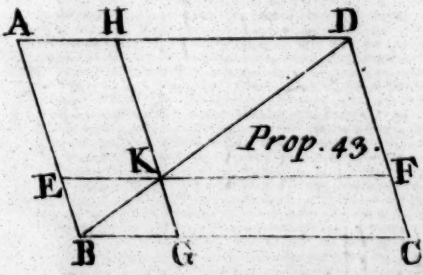
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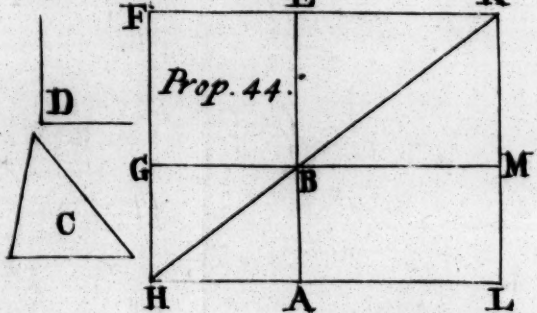
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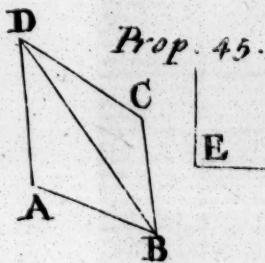
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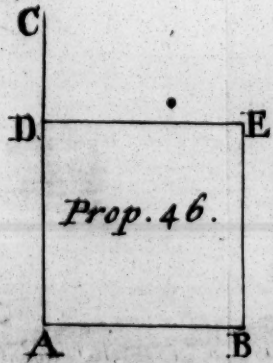
Prop. 43.



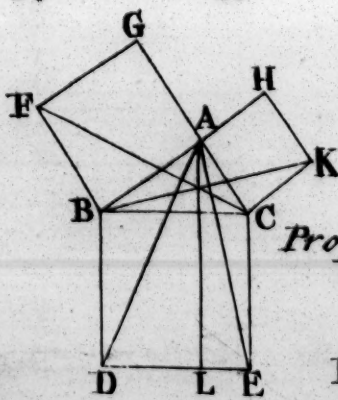
Prop. 44.



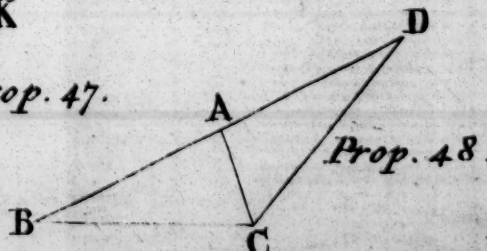
Prop. 45.



Prop. 46.



Prop. 47.



Prop. 48.



E U C L I D's ELEMENTS.

B O O K II.

DEFINITIONS.

- I. *EVERY Right-angled Parallelogram is said to be contained under two Right Lines, comprehending a Right Angle.*
- II. *In every Parallelogram, either of those Parallelograms that are about the Diameter, together with the Complements, is called a Gnomon.*

PRO.

PROPOSITION I.

THEOREM.

If there be two Right Lines, and one of them be divided into any Number of Parts; the Rectangle comprehended under the whole, and divided Line, shall be equal to all the Rectangles contained under the whole Line, and the several Segments of the divided Line.

LET A and BC be two Right Lines, whereof BC is cut or divided any how in the Points D, E. I say, the Rectangle contained under the Right Lines A and BC, is equal to the Rectangles contained under A and BD, A and DE, and A and EC.

* II. I.

† 3. I.

‡ 31. I.

For, let * BF be drawn from the Point B, at Right Angles, to BC; and make † BG equal to A; and let ‡ GH be drawn thro' G parallel to BC: Likewise let ‡ there be drawn DK, EL, CH, thro' D, E, C, parallel to BG.

Then the Rectangle BH is equal to the Rectangles BK, DL, EH; but the Rectangle BH is that contained under A and BC; for it is contained under GB, BC, and GB is equal to A; and the Rectangle BK is that contained under A and BD; for it is contained under GD and BD, and GB is equal to A; and the Rectangle DL is that contained under A and DE, because DK, that is, BG, is equal to A: So likewise the Rectangle EH is that contained under A and EC. Therefore the Rectangle under A and BC, is equal to the Rectangle under A and BD, A and DE, and A and EC. Therefore, *if there be two Right Lines given, and one of them be divided into any Number of Parts, the Rectangle comprehended under the whole, and divided Line, shall be equal to all the Rectangles contained under the whole Line, and the several Segments of the divided Line; which was to be demonstrated.*

PRO-

PROPOSITION II.

THEOREM.

If a Right Line be any-how divided, the Rectangles contained under the whole Line, and each of the Segments, or Parts, are equal to the Square of the whole Line.

LET the Right Line AB be any-how divided in the Point C. I say, the Rectangle contained under AB, BC, together with that contained under AB and AC, is equal to the Square made on AB.

For let the Square ADEB be described * on AB, * 46. 1. and thro' C let CF be drawn parallel to AD or BE. Therefore AE is equal to the Rectangles AF and CE. But AE is a Square described upon AB; and AF is the Rectangle contained under BA, AC; for it is contained under DA and AC, whereof AD is equal to AB; and the Rectangle CE is contained under AB, BC, since BE is equal to AB. Wherefore the Rectangle under AB and AC, together with the Rectangle under AB and BC, is equal to the Square of AB. Therefore, *if a Right Line be any-how divided, the Rectangles contained under the whole Line, and each of the Segments, or Parts, are equal to the Square of the whole Line.*

PROPOSITION III.

THEOREM.

If a Right Line be any-how cut, the Rectangle contained under the whole Line, and one of its Parts, is equal to the Rectangle contained under the two Parts together, with the Square of the first mentioned Part.

LET the Right Line AB be any-how cut in the Point C. I say, the Rectangle under AB and BC, is equal to the Rectangle under AC and BC, together with the Square described on BC.

* 46. I.
† 31. I.

For describe * the Square CDEB upon BC; produce ED to F; and let AF be drawn † thro' A, parallel to CD or BE.

Then the Rectangle AE shall be equal to the two Rectangles AD, CE: And the Rectangle AE is that contained under AB and BC; for it is contained under AB and BE, whereof BE is equal to BC: And the Rectangle AD is that contained under AC and CB, since DC is equal to CB: And DB is a Square described upon BC. Wherefore the Rectangle under AB and BC is equal to the Rectangle under AC and CB, together with the Square described upon BC. Therefore, *if a Right Line be any-how cut, the Rectangle contained under the whole Line, and one of its Parts, is equal to the Rectangle contained under the two Parts together, with the Square of the first-mentioned Parts*; which was to be demonstrated.

PROPOSITION IV.

THEOREM.

If a Right Line be any-how cut, the Square which is made on the whole Line, will be equal to the Squares made on the Segments thereof, together with twice the Rectangle contained under the Segments.

LET the Right Line AB be any-how cut in C. I say, the Square made on AB is equal to the Squares of AC, CB, together, with twice the Rectangle contained under AC, CB.

* 46. I.
† 31. I.

For * describe the Square ADEB upon AB, join BD, and thro' C draw † CGF parallel to AD or BE; and also thro' G draw HK parallel to AB or DE.

† 29. I.

* 5. I.

† 6. I.

† 34. I.

Then, because CF is parallel to AD, and BD falls upon them, the outward Angle BGC shall be † equal to the inward and opposite Angle ADB; but the Angle ADB is * equal to the Angle ABD, since the Side BA is equal to the Side AD. Wherefore the Angle CGB is equal to the Angle GBC; and so the Side BC equal † to the Side CG; but likewise the Side CB is † equal to the Side GK, and the Side CG to BK. Therefore GK is equal to KB, and CGKB

is equilateral. I say, it is also Right-angled; for, because CG is parallel to BK , and CB falls on them, the Angles KBC , GCB , are equal to two Right Angles. But KBC is a Right Angle. Wherefore GBC also is a Right Angle, and the opposite Angles GCB , CGK , GKB , shall be Right Angles. Therefore $CGKB$ is a Rectangle. But it has been proved to be equilateral. Therefore $CGKB$ is a Square described upon BC . For the same Reason HF is also a Square made upon HG , that is equal to the Square of AC . Wherefore HF and CK are the Squares of AC and CB . And because the Rectangle AG is * equal to the Rectangle GE , and AG is that which is contained under AC and CB ; for GC is equal to CB ; GE shall be equal to the Rectangle under AC , and CB . Wherefore the Rectangles AG , GE , are equal to twice the Rectangle contained under AC , CB ; and HF , CK , are the Squares of AC , CB . Therefore the four Figures HF , CK , AG , GE , are equal to the Squares of AC and CB , with twice the Rectangle contained under AC and CB . But HF , CK , AG , GE , make up the whole Square of AB , viz. $ADEB$. Therefore the Square of AB is equal to the Squares of AC , CB , together with twice the Rectangle contained under AC , CB . Wherefore, *if a Right Line be any-how cut, the Square which is made on the whole Line, will be equal to the Squares made on the Segments thereof, together with twice the Rectangle contained under the Segments; which was to be demonstrated.*

Coro... Hence it is manifest, that the Parallelograms which stand about the Diameter of a Square, are likewise Squares.

PROPOSITION V,

THEOREM.

If a Right Line be cut into two equal Parts, and into two unequal ones; the Rectangle under the unequal Parts, together with the Square that is made of the intermediate Distance, is equal to the Square made of half the Line.

LET any Right Line AB be cut into two equal Parts in C, and into two unequal Parts in D. I say, the Rectangle contained under AD, DB, together with the Square of CD, is equal to the Square of BC.

† 46. I.

* 31. I.

For † describe CEFB, the Square of BC; draw BE, and thro' D draw * DHG, parallel to CE, or BF; and thro' H draw KLO, parallel to EB, or EF; and AK thro' A, parallel to CL, or BO.

† 43. I.

Now the Complement CH is † equal to the Complement HF. Add DO, which is common to both of them, and the whole CO is equal to the whole DF: But CO is equal to AL, because AC is equal to CB; therefore AL is equal to DF, and adding CH, which is common, the whole AH shall be equal to FD, DL, together. But AH is the Rectangle contained under AD, DB; for DH is * equal to DB, and FD, DL, is the Gnomon MNX; therefore MNX is equal to the Rectangle contained under AD, DB; and if LG, being common, and equal to the Square of CD, be added, then the Gnomon MNX, and LG, are equal to the Rectangle contained under AD, DB, together with the Square of CD; but the Gnomon MNX, and LG, make up the whole Square CEFB, viz. the Square of CB. Therefore the Rectangle under AD, DB, together with the Square of CD, is equal to the Square of CB. Wherefore, *if a Right-Line be cut into two equal Parts, and into two unequal ones; the Rectangle under the unequal Parts, together with the Square that is made of the intermediate Distance, is equal to the Square made of half the Line; which was to be demonstrated.*

* Cor. 4. of this.

PROPOSITION VI.

THEOREM.

If a Right Line be divided into two equal Parts, and another Right Line be added directly to the same, the Rectangle contained under the Line, compounded of the whole and added Line, (taken as one Line) and the added Line, together with the Square of half the Line, is equal to the Square of the Line compounded of half the Line, and the added Line taken as one Line.

LET the Right Line AB be bisected in the Point C, and BD added directly thereto. I say, the Rectangle under AD, and DB, together with the Square of BC, is equal to the Square of CD.

For, describe * CEF D, the Square of CD, and * 46. 1. join DE; draw † BHG thro' B, parallel to CE, or † 31. 1. DF, and KLM thro' H, parallel to AD, or EF, as also AK thro' A, parallel to CL, or DM.

Then, because AC is equal to CB, the Rectangle AL shall be * equal to the Rectangle CH; but CH * 36. 1. is † equal to HF. Therefore AL will be equal to † 43. 1. HF; and adding CM, which is common to both, then the whole Rectangle AM is equal to the Gnomon NXO. But AM is that Rectangle which is contained under AD, DB; for DM is * equal to * Cor. 4. DB; therefore the Gnomon NXO is equal to of this. the Rectangle under AD, and DB. And adding LG, which is common, viz. † the Square of CB; † Cor. 4. and then the Rectangle under AD, DB, together of this. with the Square of BC, is equal to the Gnomon NXO with LG. But the Gnomon NXO, and LG, together, make up the Figure CEF D, that is, the Square of CD. Therefore the Rectangle under AD, and DB, together with the Square of BC, is equal to the Square of CD. Therefore, if a Right Line be divided into two equal Parts, and another Right Line be added directly to the same, the Rectangle contained under the Line, compounded of the whole and added Line, (taken as one Line) and the

E

added

added Line, together with the Square of half the Line, is equal to the Square of the Line compounded of half the Line, and the added Line taken as one Line; which was to be demonstrated.

PROPOSITION VII.

THEOREM.

If a Right Line be any-how cut, the Square of the whole Line, together with the Square of one of the Segments, is equal to double the Rectangle contained under the whole Line; and the said Segment, together with the Square, made of the other Segment.

LET the Right Line AB be any-how cut in the Point C . I say, the Squares of AB , BC , together, are equal to double the Rectangle contained under AB , BC , together with the Square, made of AC .

* 46. 1.

For let the Square of AB be * described, viz. $ADEB$, and construct § the Figure.

† 43. 1.

Then, because the Rectangle AG is † equal to the Rectangle GE ; if CF , which is common, be added to both, the whole Rectangle AF shall be equal to the whole Rectangle CE ; and so the Rectangles AF , CE , are double to the Rectangle AF ; but AF , CE , make up the Gnomon KLM , and the Square CF . Therefore the Gnomon KLM , together with the Square CF , shall be double to the Rectangle AF . But double the Rectangle under AB , BC , is double the Rectangle AF ; for BF is ‡ equal to BC . Therefore the Gnomon, KLM , and the Square CF , are equal to twice the Rectangle contained under AB , BC . And if HF , which is common, being the Square of AC , be added to both; then the Gnomon KLM , and the Squares CF , HF , are equal to double the Rectangle contained

† Cor. 4.

§ A Figure is said to be constructed, when Lines, drawn in a Parallelogram parallel to the Sides thereof, cut the Diameter in one Point, and make two Parallelograms about the Diameter, and two Complements. So likewise a double Figure is said to be constructed, when two Right Lines, parallel to the Sides, make four Parallelograms about the Diameter, and four Complements.

under

under AB, BC, together with the Square of AC. But the Gnomon KLM, together with the Squares CF, HF, are equal to ADEB, and CF, viz. the Squares of AB, BC. Therefore the Squares of AB, BC, are together equal to double the Rectangle contained under AB, BC, together with the Square of AC. Therefore, *if a Right Line be any-how cut, the Square of the whole Line, together with the Square of one of the Segments, is equal to double the Rectangle contained under the whole Line, and the said Segment, together with the Square, made of the other Segment;* which was to be demonstrated.

PROPOSITION VIII.

THEOREM.

If a Right Line be any-how cut into two Parts, four times the Rectangle contained under the whole Line, and one of the Parts, together with the Square of the other Part, is equal to the Square of the Line, compounded of the whole Line, and the first Part taken as one Line.

LET the Right Line AB be cut any-how in C. I say, four times the Rectangle contained under AB, BC, together with the Square of AC, is equal to the Square of AB, and BC, taken as one Line.

For, let the Right Line AB be produced to D, so that BD be equal to BC; describe the Square AEFD on AD, and construct the double Figure.

Now, since CB is * equal to BD, and also to * *Hyp.* † GK, and BD is equal to KN; GK shall be † 34. 1. likewise equal to KN; by the same Reasoning, PR is equal to RO. And since CB is equal to BD, and GK to KN, the Rectangle CK will † be † 36. 1. equal to the Rectangle BN, and the Rectangle GR to the Rectangle RN. But CK is * equal to RN; * 43. 1. for they are the Complements of the Parallelogram CO. Therefore BN is equal to GR, and the four Squares BN, KC, GR, RN, are equal to each other; and so they are together Quadruple CK. Again, because CB is equal to BD, and BD to BK, that is, equal to CG; and the said CB is equal

also to GK, that is, to GP; therefore CG shall be equal to GP. But PR is equal to RO; therefore the Rectangle AG shall be equal to the Rectangle MP, and the Rectangle PL equal to RF. But MP is equal to PL; for they are the Complements of the Parallelogram ML. Wherefore AG is equal also to RF. Therefore the four Parallelograms AG, MP, PL, RF, are equal to each other, and accordingly they are together Quadruple of AG. But it has been proved, that the four Squares CK, BN, GR, RN, are Quadruple of CK. Therefore the four Rectangles, and the four Squares making up the Gnomon STY, are together Quadruple of AK; and because AK is a Rectangle contained under AB, and BC, for BK is equal to BC; four times the Rectangle under AB, BC, will be Quadruple of AK. But the Gnomon STY has been proved to be Quadruple of AK. And so four times the Rectangle contained under AB, BC, is equal to the Gnomon STY. And if XH, being equal to † the Square of AC, which is common, be added to both; then four times the Rectangle contained under AB, BC, together with the Square of AC, is equal to the Gnomon STY, and the Square XH. But the Gnomon STY and HX make AEFD, the whole Square of AD. Therefore four times the Rectangle contained under AB, BC, together with the Square of AC, is equal to the Square of AD, that is, of AB and BC taken as one Line. Wherefore, *if a Right Line by any-how cut into two Parts, four times the Rectangle contained under the whole Line, and one of the Parts, together with the Square of the other Part, is equal to the Square of the Line, compounded of the whole Line, and the first Part taken as one Line; which was to be demonstrated.*

† Cor. 4.
of this.

PROPOSITION IX.

THEOREM.

If a Right Line be any-how cut into two equal, and two unequal Parts; then the Squares of the unequal Parts together, are double to the Square of the half Line, and the Square of the intermediate Part.

LET any Right Line AB be cut unequally in D, and equally in C. I say, the Squares of AD, DB, together, are double to the Squares of AC and CD together.

For, let * CE be drawn from the Point C at Right ^{* 11. 1.} Angles to AB, which make equal to AC, or CB; and join EA, EB. Also thro' D let † DF be drawn † ^{31. 1.} parallel to CE, and FG thro' F parallel to AB, and draw AF.

Now, because AC is equal to CE, the Angle EAC will be † equal to the Angle AEC; and since the † ^{5. 1.} Angle at C is a Right one, the other Angles, AEC, EAC, together, shall * make one Right Angle, and ^{* 3. Cor.} are equal to each other: And so AEC, EAC, are each ^{32. 1.} equal to half a Right Angle. For the same Reason are also CEB, EBC, each of them half Right Angles. Therefore the whole Angle AEB is a Right Angle. And since the Angle GEF is half a Right one, and EGF is a Right Angle; for it is † equal to the † ^{29. 1.} inward and opposite Angle ECB; the other Angle EFG will be also equal to half a Right one. Therefore the Angle GEF is equal to the Angle EFG. And so the Side EG is † equal to the Side GF. Again, † ^{6. 1.} because the Angle at B is half a Right one, and FDB is a Right one, because equal to the inward and opposite Angle ECB, the other Angle BFD will be half a Right Angle. Therefore the Angle at B is equal to the Angle BFD; and so the Side DF is equal to the Side DB. And because AC is equal to CE, the Square of AC will be equal to the Square of CE. Therefore the Squares of AC, CE, together, are double to the Square of AC; but the Square of EA is † equal to the Squares of AC, CE, together, since † ^{47. 1.}

E 3

ACE

† 47. 1.

ACE is a Right Angle. Therefore the Square of EA is double to the Square of AC. Again, because EG is equal to GF, and the Square of EG is equal to the Square of GF; therefore the Squares of EG, GF, together, are double to the Square of GF. But the Square of EF is † equal to the Squares of EG, GF. Therefore the Square of EF is double the Square of GF: But GF is equal to CD; and so the Square of EF double to the Square of CD. But the Square of AE is likewise double to the Square of AC. Wherefore the Squares of AE, and EF, are double to the Squares of AC, and CD. But the Square of AF is † equal to the Squares of AE and EF; because the Angle AEF is a Right Angle, and consequently the Square of AF is double to the Squares of AC, and CD. But the Squares of AD, DF, are equal to the Square of AF: For the Angle at D is a Right Angle. Therefore the Square of AD and DF, together, shall be double to the Squares of AC and CD together. But DF is equal to DB. Therefore the Squares of AD, and DB, together, will be double to the Squares of AC and CD, together. Wherefore, *if a Right Line be any-how cut into two equal, and two unequal Parts; then the Squares of the unequal Parts together are double to the Square of the half Line, and the Square of the intermediate Part; which was to be demonstrated.*

PROPOSITION X.

THEOREM.

If a Right Line be cut into two equal Parts, and to it be directly added another; the Square made on [the Line compounded of] the whole Line, and the added one, together with the Square of the added Line, shall be double to the Square of the half Line, and the Square of [that Line which is compounded of] the half, and the added Line.

LET the Right Line AB be bisected in C, and any straight Line BD added directly thereto. I say, the Squares of AD, DB, together, are double to the Squares of AC, CD, together.

For,

For, draw * CE from the Point C at Right Angles * 11. 1. to AB, which make equal to AC, or CB, and draw AE, EB; likewise thro' E let EF be † drawn parallel † 31. 1. to AD, and thro' D, DF † parallel to CE.

Then, because the Right Line EF falls upon the Parallels EC, FD, the Angles CEF, EFD, are † equal † 29. 1. to two Right Angles. Therefore the Angles FEB, EFD, are together less than two Right Angles. But Right Lines making, with a third Line, Angles together less than two Right Angles, being infinitely produced, will meet *. Wherefore EB, FD, pro- * Ax. 12. duced, will meet towards BD. Now let them be produced, and meet each other in the Point G, and let AG be drawn.

And then, because AC is equal to CE, the Angle AEC will be equal to the Angle EAC †: But the † 5. 1. Angle at C is a Right Angle. Therefore the Angle CAE, or AEC, is half a Right one. By the same way of Reasoning, the Angle CEB, or EBC, is half a Right one. Therefore AEB is a Right Angle. And since EBC is half a Right Angle, DBG will † also † 15. 1. be half a Right Angle, since it is vertical to CBE. But BDG is a Right Angle also; for it is * equal to * 29. 1. the alternate Angle DCE. Therefore the remaining Angle DGB is half a Right Angle, and so equal to DBG. Wherefore the Side BD is † equal to the † 6. 1. Side DG. Again, because EGF is half a Right Angle, and the Angle at F is a Right Angle, for it is equal to the opposite Angle at C; the remaining Angle FEG will be also half a Right one, and is equal to the Angle EGF; and so the Side GF is † equal to the Side EF. And since EC is equal to CA, and the Square of EC equal to the Square of CA; therefore the Squares of EC, CA, together, are double to the Square of CA. But the Square of EA is † equal † 47. 1. to the Squares of EC, CA. Wherefore the Square of EA is double to the Square of AC. Again, because GF is equal to FE, the Square of GF also is equal to the Square of FE. Wherefore the Squares of GF, FE, are double to the Square of FE. But the Square of EG is † equal to the Squares of GF, FE. Therefore the Square of EG is double to the Square of EF: But EF is equal to CD. Wherefore the Square of EG shall be double to the Square of CD.

E 4

But

For,

But the Square of EA has been proved to be double to the Square of AC. Therefore the Squares of AE, EG, are double the Squares of AC, CD. But the Square of AG is † equal to the Squares of AE, EG; and consequently the Square of AG is double to the Squares of AC, CD. But the Squares of AD, DG, are † equal to the Square AG. Therefore the Squares of AD, DG, are double to the Squares of AC, CD. But DG is equal to DB. Wherefore the Squares of AD, DB, are double to the Squares of AC, CD. Therefore, *if a Right Line be cut into two equal Parts, and to it be directly added another; the Square made on [the Line compounded of] the whole Line, and the added one, together with the Square of the added Line, shall be double to the Square of the half Line, and the Square of [that Line which is compounded of] the half, and the added Line.*

PROPOSITION XI.

PROBLEM.

To cut a given Right Line so, that the Rectangle contained under the whole Line, and one Segment, be equal to the Square of the other Segment.

LET AB be a given Right Line. It is required to cut the same so, that the Rectangle contained under the Whole, and one Segment thereof, be equal to the Square of the other Segment.

* 46. I. Describe * ABCD the Square of AB; bisect AC in E, and draw BE: Also produce CA to F, so that EF be equal to EB. Describe FGHA the Square of AF, and produce GH to K. I say, AB is cut in H so, that the Rectangle under AB, BH, is equal to the Square of AH.

For, since the Right Line AC is bisected in E, and AF is directly added thereto, the Rectangle under CF, FA, together with the Square of AE, will be † equal to the Square of EF. But EF is equal to EB. Therefore the Rectangle under CF, FA, together with the Square of AE, is equal to the Square of EB. But the Squares of BA, AE, are † equal to the Square of EB; for the Angle at A is a Right Angle.

† 6 of this.

† 47. I.

Angle. Therefore the Rectangle under CF, FA, together with the Square of AE, is equal to the Squares of BA, AE. And, taking away the Square of AE, which is common, the remaining Rectangle under CF, FA, is equal to the Square of AB. But FK is the Rectangle under CF, FA; since AF is equal to FG; and the Square of AB is AD. Wherefore the Rectangle FK is equal to the Square AD. And if AK, which is common, be taken from both, then the remaining Square FH is equal to the remaining Rectangle HD. But HD is the Rectangle under AB, BH, since AB is equal to BD, and FH is the Square of AH. Therefore the Rectangle under AB, BH, shall be equal to the Square of AH. And so the given Right Line AB is cut in H, so that the Rectangle under AB, BH, is equal to the Square of AH; *which was to be done.*

PROPOSITION XII.

THEOREM.

In obtuse-angled Triangles, the Square of the Side subtending the obtuse Angle is greater than the Squares of the Sides containing the obtuse Angle, by twice the Rectangle under one of the Sides, containing the obtuse Angle, viz. that on which, produced, the Perpendicular falls, and the Line taken without, between the Perpendicular, and the obtuse Angle.

LET ABC be an obtuse-angled Triangle, having the obtuse Angle BAC; and * from the Point B * 12. 1. draw BD perpendicular to the Side CA produced. I say, the Square of BC is greater than the Squares of BA and AC, by twice the Rectangle contained under CA, and AD.

For, because the Right Line CD is any-how cut in the Point A, the Square of CD shall be † equal to the † 4 of this. Squares of CA, and AD, together with twice the Rectangle under CA, and AD. And if the Square of BD, which is common, be added, then the Squares of CD, DB, are equal to the Squares of CA, AD, and

* 47. I.

and DB, and twice the Rectangle contained under CA and AD. But the Square of CB is * equal to the Squares of CD, DB; for the Angle at D is a Right one, since BD is perpendicular, and the Square of AB is * equal to the Squares of AD, and DB. Therefore the Square of CB is equal to the Squares of CA and AB, together with twice a Rectangle under CA and AD. Therefore *in obtuse-angled Triangles, the Square of the Side subtending the obtuse Angle, is greater than the Squares of the Sides containing the obtuse Angle, by twice the Rectangle under one of the Sides containing the obtuse Angle, viz. that on which, produced, the Perpendicular falls, and the Line taken without, between the Perpendicular and the obtuse Angle; which was to be demonstrated.*

PROPOSITION XIII.

THEOREM.

In acute-angled Triangles, the Square of the Side subtending the acute Angles, is less than the Squares of the Sides containing the acute Angle, by twice a Rectangle under one of the Sides about the acute Angle, viz. on which the Perpendicular falls, and the Line assumed within the Triangle, from the Perpendicular to the acute Triangle.

* 12. I.

LET ABC be an acute-angled Triangle, having the acute Angle B; and from A let there * be drawn AD perpendicular to BC. I say, the Square of AC is less than the Squares of CB and BA, by twice a Rectangle under CB and BD.

† 7 of this.

For, because the Right Line CB is cut any-how in D, the Squares of CB and BD will be † equal to twice a Rectangle under CB and BD, together with the Square of DC. And if the Square of AD be added to both, then the Squares of CB, BD, and DA, are equal to twice the Rectangle contained under CB and BD, together with the Squares of AD and DC. But the Square of AB is † equal to the Squares of BD and DA; for the Angle at D is a Right Angle. And the Square of AC is † equal to the

† 47. I.

the Squares of AD and DC. Therefore the Squares of CB and BA are equal to the Square of AC, together, with twice the Rectangle contained under CB and BD. Wherefore the Square of AC only, is less than the Squares of CB and BA, by twice the Rectangle under CB and BD. Therefore *in acute-angled Triangles, the Square of the Side subtending the acute Angles, is less than the Squares of the Sides containing the acute Angle, by twice a Rectangle under one of the Sides about the acute Angle, viz. on which the Perpendicular falls, and the Line assumed within the Triangle from the Perpendicular to the acute Angle; which was to be demonstrated.*

PROPOSITION XIV.

PROBLEM.

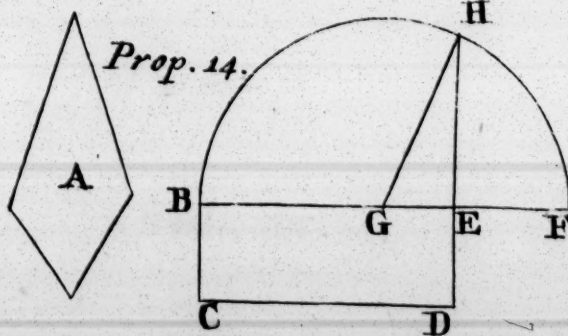
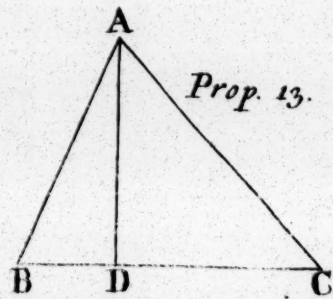
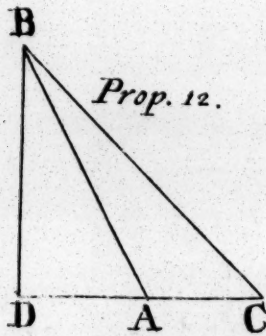
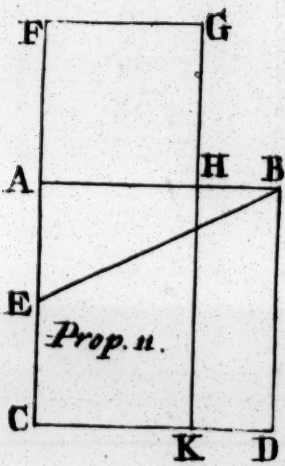
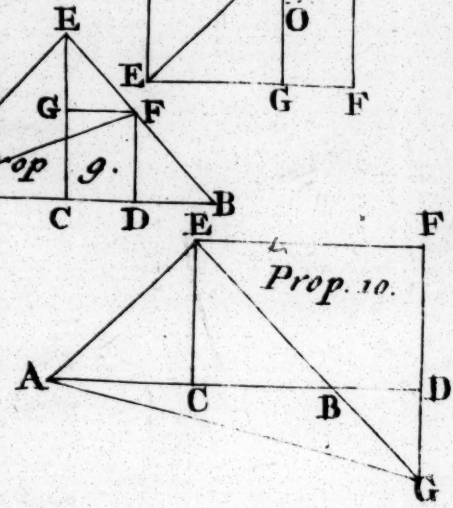
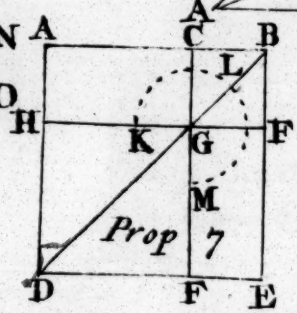
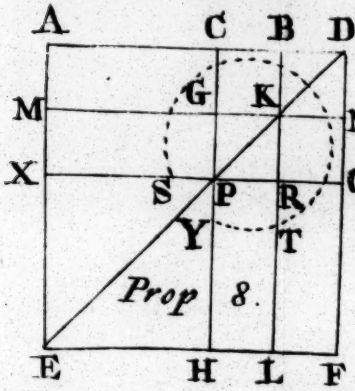
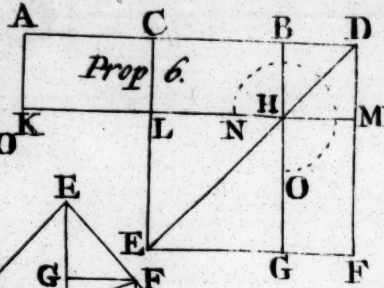
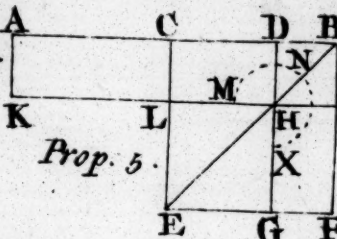
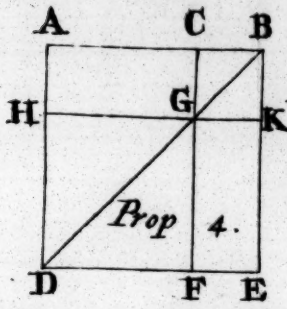
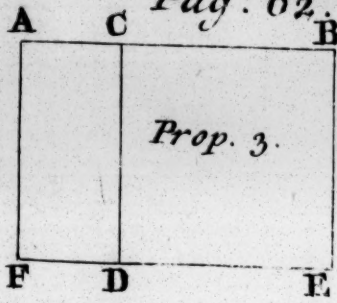
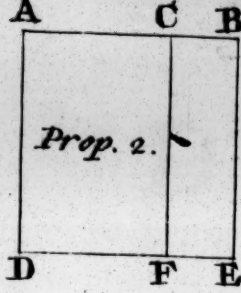
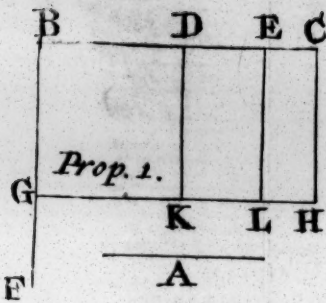
To make a Square equal to a given Right-lined Figure.

LET A be the given Right-lined Figure. It is required to make a Square equal thereto.

Make * the Right-angled Parallelogram BCDE * 45. 1. equal to the Right-lined Figure A. Now if BE be equal to ED, what was proposed will be already done, since the Square BD is made equal to the Right-lined Figure A: But if it be not, let either BE or ED be the greater: Suppose BE, which let be produced to F; so that EF be equal to ED. This being done, let BE be † bisected in G, about which, as a Centre, † 10. 1. with the Distance GB or GF, describe the Semicircle BHF; and let DE be produced to H, and draw GH. Now, because the Right Line BF is divided into two equal Parts in G, and into two unequal ones in E, the Rectangle under BE and EF, together with the Square of GE, shall be ‡ equal to the Square of GF. ‡ 5 of this. But GF is equal to GH. Therefore the Rectangle under BE, EF, together, with the Square of GE, is equal to the Square of GH. But the Squares of HE and EG are * equal to the Square of GH. * 47. 1. Wherefore the Rectangle under BE, EF, together with the Square of EG, is equal to the Squares of HE, EG. And if the Square of EG, which is common,

mon, be taken from both, the remaining Rectangle, contained under BE and EF , is equal to the Square of EH . But the Rectangle under BE and EF is the Parallelogram BD , because EF is equal to ED . Therefore the Parallelogram BD is equal to the Square of EH ; but the Parallelogram BD is equal to the Right-lin'd Figure A . Wherefore the Right-lined Figure A is equal to the Square of EH . And so there is a Square made equal to the given Right-lined Figure A , viz. the Square of EH ; which was to be done.

The END of the SECOND BOOK.





E U C L I D's ELEMENTS.

B O O K III.

DEFINITIONS.

- I. **E**QUAL Circles are such whose Diameters are equal; or from whose Centres the Right Lines that are drawn are equal.
- II. A Right Line is said to touch a Circle, when touching the same, and being produced, does not cut it.
- III. Circles are said to touch each other, which touching do not cut one another.
- IV. Right Lines in a Circle are said to be equally distant from the Centre, when Perpendiculars drawn from the Centre to them be equal.
- V. And that Line is said to be farther from the Centre, on which the greater Perpendicular falls.
- VI. A Segment of a Circle is a Figure contained under a Right Line, and a Part of the Circumference of a Circle.
- VII. An Angle of a Segment is that which is contained by a Right Line, and the Circumference of a Circle.

VIII. An

VIII. *An Angle is said to be in a Segment, when some Point is taken in the Circumference thereof, and from it Right Lines are drawn to the Ends of that Right Line, which is the Base of the Segment; then the Angle contained under the Lines drawn, is said to be an Angle in a Segment.*

IX. *But when the Right Lines containing the Angle do receive any Circumference of the Circle, then the Angle is said to stand upon that Circumference.*

X. *A Sector of a Circle is that Figure comprehended between the Right Lines drawn from the Centre, and the Circumference contained between them.*

XI. *Similar Segments of Circles are those which include equal Angles, or whereof the Angles in them are equal.*

PROPOSITION I.

PROBLEM.

To find the Centre of a Circle given.

LET ABC be the Circle given. It is required to find the Centre thereof.

Let the Right Line AB be any-how drawn in it, which * bisect in the Point D; and let DC be † drawn from the Point D at Right Angles to AB, which let be produced to E.

* 10. 1.

† 11. 1.

Then, if EC be * bisected in F, I say, the Point F is the Centre of the Circle ABC.

For, if it be not, let G be the Centre, and let GA, GD, GB, be drawn. Now, because DA is equal to DB, and DG is common, the two Sides AD, DG, are equal to the two Sides GD, DB, each to each; and the Base GA is † equal to the Base GB; for they are drawn from the Centre G. Therefore the Angle

† Def. 15. 1.

* 8. 1.

ADG is * equal to the Angle GDB. But when a Right Line standing upon a Right Line, makes the adjacent Angles equal to one another, each of the equal Angles will † be a Right Angle. Wherefore the Angle GDB is a Right Angle. But FDB is also a

† Def. 10. 1.

Right,

Right Angle. Therefore the Angle FDB is equal to the Angle GDB , a greater to a less, which is absurd. Wherefore G is not the Centre of the Circle ABC . After the same manner we prove, that no other Point, unless F , is the Centre. Therefore F is the Centre of the Circle ABC ; *which was to be found.*

Coroll. If in a Circle, any Right Line cuts another Right Line into two equal Parts, and at Right Angles; the Centre of the Circle will be in that cutting Line.

PROPOSITION II.

THEOREM.

If any two Points be assumed in the Circumference of a Circle, the Right Line joining those two Points shall fall within the Circle.

LET ABC be a Circle; in the Circumference of which let any two Points A, B , be assumed. I say, a Right Line drawn from the Point A to the Point B , falls within the Circle.

For let any Point E be taken in the Right Line AB , and let DA, DE, DB , be joined.

Then, because DA is equal to DB , the Angle DAB will be * equal to the Angle DBA ; and since the * 5. 1. Side AE of the Triangle DAE is produced, the Angle DEB will be † greater than the Angle DAE . † 16. 1. But the Angle DAE is equal to the Angle DBE ; therefore the Angle DEB is greater than the Angle DBE . But the greater Side subtends the greater Angle. Wherefore DB is greater than DE . But DB only comes to the Circumference of the Circle; therefore DE does not reach so far. And so the Point E falls within the Circle. Therefore, *if two Points are assumed in the Circumference of a Circle, the Right Line joining those two Points shall fall within the Circle; which was to be demonstrated.*

Coroll. Hence, if a Right Line touches a Circle, it will touch it in one Point only.

PROPOSITION III.

THEOREM.

If in a Circle a Right Line drawn thro' the Centre cuts any other Right Line not drawn thro' the Centre, into equal Parts, it shall cut it at Right Angles; and if it cuts it at Right Angles, it shall cut it into two equal Parts.

LET ABC be a Circle, wherein the Right Line CD, drawn thro' the Centre, bisects the Right Line AB not drawn thro' the Centre. I say, it cuts it at Right Angles.

* 1 of this. For, * find E the Centre of the Circle, and let EA, EB, be joined.

Then, because AF is equal to FB, and FE is common, the two Sides AF, FE, are equal to the two Sides BF, FE, each to each; but the Base EA is equal to the Base EB. Wherefore the Angle AFE shall be † equal to the Angle BFE. But when a Right Line standing upon a Right Line makes the adjacent Angles equal to one another, each of the equal Angles is † a Right Angle. Wherefore AFE, or BFE, is a Right Angle. And therefore the Right Line CD drawn thro' the Centre, bisecting the Right Line AB not drawn thro' the Centre, cuts it at Right Angles. Now if CD cuts AB at Right Angles, I say, it will bisect it, that is, AF will be equal to FB. For the same Construction remaining, because EA, being drawn from the Centre, is equal to EB, the Angle EAF shall be * equal to the Angle EBF. But the Right Angle AFE is equal to the Right Angle BFE; therefore the two Triangles EAF, EBF, have two Angles of the one equal to two Angles of the other, and the Side EF is common to both. Wherefore the other Sides of the one shall be † equal to the other Sides of the other: And so AF will be equal to FB. Therefore, *if in a Circle a Right Line drawn thro' the Centre cuts any other Right Line not drawn thro' the Centre, into two equal Parts, it shall cut it at Right Angles; and if it cuts it at Right Angles, it shall cut it into two equal Parts; which was to be demonstrated.*

P R O

PROPOSITION IV.

THEOREM.

If in a Circle two Right Lines, not being drawn thro' the Centre, cut each other, they will not cut each other into two equal Parts.

LET $ABCD$ be a Circle, wherein two Right Lines AC , BD , not drawn thro' the Centre, cut each other in the Point E . I say, they do not bisect each other.

For, if possible, let them bisect each other, so that AE be equal to EC , and BE to ED . Let the Centre F of the Circle $ABCD$ be * found, and join EF . * 1 of this.

Then, because the Right Line FE , drawn thro' the Centre, bisects the Right Line AC , not drawn thro' the Centre, it will † cut AC at Right Angles, And † 3 of this. so FEA is a Right Angle. Again, because the Right Line FE bisects the Right Line BD not drawn thro' the Centre, it will † cut BD at Right Angles. Therefore FEB is a Right Angle. But FEA has been shewn to be also a Right Angle. Wherefore the Angle FEA will be equal to the Angle FEB , a less to a greater; which is absurd. Therefore AC , BD , do not mutually bisect each other. And so, *if in a Circle two Right Lines, not being drawn thro' the Centre, cut each other, they will not cut each other into two equal Parts; which was to be demonstrated.*

PROPOSITION V.

THEOREM.

If two Circles cut one another, they shall not have the same Centre.

LET the two Circles ABC , CDG , cut each other in the Points B , C . I say, they have not the same Centre.

For, if they have, let it be E , and join EC , and draw EF G at Pleasure.

F

Now,

Now, because E is the Centre of the Circle ABC, CE will be equal to EF. Again, because E is the Centre of the Circle CDG, CE is equal to EG. But CE has been shewn to be equal to EF. Therefore EF shall be equal to EG, a less to a greater, which cannot be. Therefore the Point E is not the Centre of both the Circles ABC, CDG. Wherefore, *if two Circles cut one another, they shall not have the same Centre*; which was to be demonstrated.

PROPOSITION VI.

THEOREM.

If two Circles touch one another inwardly, they will not have one and the same Centre.

LET two Circles ABC, CDE, touch one another inwardly in the Point C. I say, they will not have one and the same Centre.

For, if they have, let it be F, and join FC, and draw FB any-how.

Then, because F is the Centre of the Circle ABC, CF is equal to FB. And because F is only the Centre of the Circle CDE, CF shall be equal to EF. But CF has been shewn to be equal to FB. Therefore FE is equal to FB, a less to a greater; which cannot be. Therefore the Point F is not the Centre of both the Circles ABC, CDE. Wherefore, *if two Circles touch one another inwardly, they will not have one and the same Centre*; which was to be demonstrated.

PROPOSITION VII.

THEOREM.

If in the Diameter of a Circle some Point be taken, which is not the Centre of the Circle, and from that Point certain Right Lines fall on the Circumference of the Circle, the greatest of these Lines shall be that wherein the Centre of the Circle is; the least, the Remainder of the same Line. And of all the other Lines, the nearest to that which was drawn thro' the Centre, is always greater than that more remote, and only two equal Lines fall from the abovesaid Point upon the Circumference, on each Side of the least or greatest Lines.

LET $ABCD$ be a Circle, whose Diameter is AD , in which assume some Point F , which is not the Centre of the Circle. Let the Centre of the Circle be E ; and from the Point F let certain right Lines FB, FC, FG , fall on the Circumference: I say, FA is the greatest of these Lines, and FD the least; and of the others FB is greater than FC , and FC greater than FG .

For let BE, CE, GE , be join'd.

Then, because two Sides of every Triangle are * greater than the third; BE, EF , are greater than * 20. 1. BF . But AE is equal to BE . Therefore BE and EF are equal to AF . And so AF is greater than FB .

Again, because BE is equal to CE , and FE is common, the two Sides BE and FE are equal to the two Sides CE, EF . But the Angle BEF is greater than the Angle CEF . Wherefore the Base BF is greater than the Base FC †. For the same † 24. 1. Reason, CF is greater than FG .

Again, because GF and FE are * greater than * 20. 1. GE ; and GE is equal to ED ; GF and FE shall be greater than ED ; and if FE , which is common, be taken away, then the Remainder GF is greater than the Remainder FD . Wherefore FA is the

greatest of the Right Lines, and FD the least: Also BF is greater than FC ; and FC greater than FG .

I say, moreover, that there are only two equal Right Lines, that can fall from the Point F on $ABCD$, the Circumference of the Circle on each Side the shortest Line FD . For at the given Point E , with the Right Line EF , make † the Angle FEH equal to the Angle GEF , and join FH . Now because GE is equal to EH , and EF is common, the two Sides GE and EF are equal to the two Sides HE and EF . But the Angle GEF is equal to the Angle HEF . Therefore the Base FG shall be † equal to the Base FH . I say, no other Right Line falling from the Point F , on the Circle, can be equal to FG . For if there can, let this be FK . Now since FK is equal to FG , as also FH , FK , will be equal to FH , viz. a Line drawn nigher to that passing thro' the Centre, equal to one more remote, which cannot be. *If, therefore, in the Diameter of a Circle, some Point be taken, which is not the Centre of the Circle, and from that Point certain Right Lines fall on the Circumference of the Circle, the greatest of these Lines shall be that wherein the Centre of the Circle is; the least, the Remainder of the same Line. And of all the other Lines, the nearest to that which was drawn through the Centre, is always greater than that more remote; and only two equal Lines fall from the abovesaid Point upon the Circumference, on each Side of the least or greatest Lines; which was to be demonstrated.*

PROPOSITION VIII.

THEOREM.

If some Point be assumed without a Circle, and from it certain Right Lines be drawn to the Circle, one of which passes thro' the Centre, but the other any-how; the greatest of these Lines, is that passing thro' the Centre, and falling upon the concave Part of the Circumference of the Circle; and of the others, that which is nearest to the Line passing thro' the Centre is greater than that more remote. But the least of the Lines that fall upon the convex Circumference of the Circle, is that which lies between the Point and the Diameter; and of the others, that which is nigher to the least, is less than that which is further distant; and from that Point there can be drawn only two equal Lines, which shall fall on the Circumference on each Side the least Line.

LET ABC be a Circle, out of which take any Point D . From this Point let there be drawn certain Right Lines DA, DE, DF, DC , to the Circle, whereof DA passes thro' the Centre. I say, DA , which passes thro' the Centre, is the greatest of the Lines falling upon AEC , the concave Circumference of the Circle; and the least is DG , viz. the Line drawn from D to the Diameter GA : Likewise DE is greater than DF , and DF greater than DC . But of these Lines that fall upon $HLGK$ the convex Circumference of the Circle, that which is nearest the least DG , is always less than that more remote; that is, DK is less than DL , and DL less than DH .

For, find * M the Centre of the Circle ABC , * 1 of this, and let ME, MF, MC, MH, ML , be joined.

Now, because AM is equal to ME ; if MD , which is common, be added, AD will be equal to EM and MD . But EM and MD are † greater † 20. than ED ; therefore AD is also greater than ED .

F 3

Again,

Again, because ME is equal to MF , and MD is common, then EM , MD , shall be equal to MF , MD ; and the Angle EMD is greater than the Angle FMD . Therefore the Base ED will be † greater than the Base FD . We prove, in the same manner, that FD is greater than CD . Wherefore DA is the greatest of the right Lines falling from the Point D ; DE is greater than DF , and DF is greater than DC .

Moreover, because MK and KD are * greater than MD , and MG is equal to MK ; then the Remainder KD will † be greater than the Remainder GD . And so GD is less than KD , and consequently is the least. And because two right Lines MK , KD , are drawn from M and D to the Point K , within the Triangle MLD , MK , and KD , are † less than ML , and LD ; but MK is equal to ML . Wherefore the Remainder DK is less than the Remainder DL . In like manner we demonstrate, that DL is less than DH . Therefore DG is the least. And DK is less than DL , and DL than DH .

I say, likewise, that from the Point D only two equal right Lines can fall upon the Circle on each Side the least Line. For, make * the Angle DMB at the Point M , with the right Line MD , equal to the Angle KMD , and join DB . Then, because MK is equal to MB , and MD is common, the two Sides KM , MD , are equal to the two Sides BM , MD , each to each; but the Angle KMD is equal to the Angle BMD . Therefore the Base DK is † equal to the Base DB . I say, no other Line can be drawn from the Point D to the Circle equal to DK ; for if there can, let DN . Now, since DK is equal to DN , as also to DB , therefore DB shall be equal to DN , viz. the Line drawn nearest to the least equal to that more remote, which has been shewn to be impossible. Therefore, if some Point be assum'd without a Circle, and from it certain right Lines be drawn to the Circle, one of which passes through the Centre, but the others any-how; the greatest of these Lines is that passing through the Centre, and falling upon the concave Part of the Circumference of the Circle; and of the others, that which is nearest to the Line passing through the Centre, is greater than that more

more remote. But the least of the Lines that fall upon the convex Circumference of the Circle, is that which lies between the Point and the Diameter; and of the others, that which is nigher to the least, is less than that which is farther distant; and from that Point there can be drawn only two equal Lines, which shall fall on the Circumference on each Side the least Line; which was to be demonstrated.

PROPOSITION IX.

THEOREM.

If a Point be assumed in a Circle, and from it more than two equal Right Lines be drawn to the Circumference; then that Point is the Centre of the Circle.

LET the Point D be assum'd within the Circle ABC; and from the Point D, let there fall more than two equal right Lines to the Circumference, viz. the right Lines DA, DB, DC. I say, the assum'd Point D is the Centre of the Circle ABC.

For, if it be not, let E be the Centre, if possible; and join DE, which produce to G and F.

Then FG is a Diameter of the Circle ABC; and so, because the Point D, not being the Centre of the Circle, is assum'd in the Diameter FG, DG will * be the greatest Line drawn from D to the Circumference, and DC greater than DB, and DB than DA; but they are also equal, which is absurd. Therefore E is not the Centre of the Circle ABC. And in this manner we prove, that no other Point, except D, is the Centre; therefore D is the Centre of the Circle ABC; which was to be demonstrated.

Otherwise:

Let ABC be the Circle, within which take the Point D, from which let more than two equal right Lines fall on the Circumference of the Circle, viz. the three equal ones DA, DB, DC: I say, the Point D is the Centre of the Circle ABC.

F 4

For,

*10. 1. Euc.

*8. 1.

For, join AB , BC , which bisect * in the Points E and Z ; as also join ED , DZ ; which produce to the Points H , K , O , L ; then, because AE is equal to EB , and ED is common, the two Sides AE , ED , shall be equal to the two Sides BE , ED . And the Base DA is equal to the Base DB : Therefore the Angle AED will be * equal to the Angle BED : and so [by *Def.* 10. 1.] each of the Angles AED , BED , is a Right Angle: Therefore HK , bisecting AB , cuts it at Right Angles. And because a Right Line in a Circle, bisecting another Right Line, cuts it at Right Angles, and the Centre of the Circle is in the cutting Line, [by *Cor.* 1. 3.] the Centre of the Circle ABC will be in HK . For the same Reason, the Centre of the Circle will be in OL . And the Right Lines HK , OL , have no other Point common but D : Therefore D is the Centre of the Circle ABC ; which was to be demonstrated.

PROPOSITION X.

THEOREM.

A Circle cannot cut another Circle in more than two Points.

FOR, if it can, let the Circle ABC cut the Circle DEF in more than two Points, viz. in B , G , F ; and let K be the Centre of the Circle ABC , and join KB , KG , KF .

Now, because the Point K is assumed within the Circle DEF , from which more than two equal Right Lines KB , KG , KF , fall on the Circumference, the Point K shall be † the Centre of the Circle DEF .
 † By Hyp. But K is ‡ the Centre of the Circle ABC . Therefore K will be the Centre of two Circles cutting each other, which is absurd. Wherefore a Circle cannot cut a Circle in more than two Points; which was to be demonstrated.

PROPOSITION XI.

THEOREM.

If two Circles touch each other on the Inside, and the Centres be found, the Line joining their Centres will fall on the [Point of] Contact of those Circles.

LET two Circles ABC , ADE , touch one another inwardly in A , and let F be the Centre of the Circle ABC , and G that of ADE . I say, a right Line joining the Centres G and F , being produced, will fall in the Point A .

If this be denied, let the right Line, joining FG , cut the Circle in D and H .

Now, because AG , GF , are greater than AF , * that is, than FH ; take away FG , which is common, and the Remainder AG is greater than the Remainder GH . But AG is equal to GD ; therefore GD is greater than GH , the less than the greater, which is absurd. Wherefore a Line drawn thro' the Points F , G , will not fall out of the Point of Contact A , and so necessarily must fall in it; which was to be demonstrated.

PROPOSITION XII.

THEOREM.

If two Circles touch one another on the Outside, a Right Line joining their Centres will pass thro' the [Point of] Contact.

LET two Circles ABC , ADE , touch one another outwardly in the Point A ; and let F be the Centre of the Circle ABC , and G that of ADE . I say, a right Line drawn through the Centre F , G , will pass thro' the Point of Contact A .

For, if it does not, let, if possible, $FCDG$ fall without it, and join FA , AG .

Now, since F is the Centre of the Circle ABC , AF will be equal to FC . And because G is the Centre of the

* 20. I.

the Circle ADE, AG will be equal to GD: But AF has been shewn to be equal to FC; therefore FA, AG, are equal to FC, DG. And so the Whole FG is greater than FA, AG; and also less, * which is absurd. Therefore a right Line, drawn from the Point F to G, will pass through the Point of Contact A; which was to be demonstrated.

PROPOSITION XIII.

THEOREM.

One Circle cannot touch another in more Points than one, whether it be inwardly or outwardly.

FOR, in the first Place, if this be denied, let the Circle ABDC, if possible, touch the Circle EBFD inwardly, in more Points than one, viz. in BD.

And let G be the Centre of the Circle ABDC, and H that of EBFD.

Then a right Line, drawn from the Point G to H, will † fall in the Points B and D. Let this Line be BGHD. And because G is the Centre of the Circle ABDC, the Line BG will be equal to GD. Therefore BG is greater than HD, and BH much greater than HD. Again, since H is the Centre of the Circle EBFD, the Line BH is equal to HD. But it has been prov'd to be much greater than it, which is absurd. Therefore one Circle cannot touch another Circle inwardly in more Points than one.

Secondly, Let the Circle ACK, if possible, touch the Circle ABDC outwardly in more Points than one, viz. in A and C; and let A, C, be join'd.

Now, because two Points A, C, are assum'd in the Circumference of each of the Circles ABDC, ACK, a right Line joining these two Points, will fall ‡ within either of the Circles. But it falls within the Circle ABDC, and without the Circle ACK, which is absurd. Therefore one Circle cannot touch another Circle in more Points than one outwardly. But it has been prov'd, that one Circle cannot touch another Circle inwardly [in more Points than one]. Wherefore one Circle cannot touch another in more Points

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Points than one, whether it be inwardly or outwardly; which was to be demonstrated.

PROPOSITION XIV.

THEOREM.

Equal Right Lines in a Circle are equally distant from the Centre; and Right Lines, which are equally distant from the Centre, are equal between themselves.

LET $ABDC$ be a Circle, wherein are the equal right Lines AB , CD . I say, these Lines are equally distant from the Centre of the Circle.

For, let E be the Centre of the Circle $ABDC$; from which let there be drawn EF , FG , perpendicular to AB , CD ; and let AE , EC , be join'd.

Then, because a right Line EF , drawn thro' the Centre, cuts the right Line AB , not drawn thro' the Centre, at right Angles, it will * bisect the same. * *3 of this.*

Wherefore AF is equal to FB , and so AB is double to AF . For the same Reason CD is double to CG , but AB is equal to CD . Therefore AF is equal to CG ; and because AE is equal to EC , the Square of AE shall be equal to the Square of EC . But the Squares of AF and FE are † equal to the Square of † *47. 1.*

AE . For the Angle at F is a right Angle; and the Squares of EG , and GC , are equal to the Square of EC , since the Angle at G is a right one. Therefore the Squares of AF , and FE , are equal to the Squares of CG , and GE . But the Square of AF is equal to the Square of CG ; for AF is equal to CG . Therefore the Square of FE is equal to the Square of EG ; and so FE equal to EG . Also Lines in a Circle are ‡ said to be equally distant from ‡ *Def. 4. of this.* the Centre, when Perpendiculars drawn to them from the Centre are equal. Therefore AB , CD , are equally distant from the Centre.

But if AB , CD , are equally distant from the Centre, that is, if FE be equal to EG ; I say, AB is equal to CD .

For the same Construction being suppos'd, we demonstrate, as above, that AB is double to AF , and CD

† 47. 1.

CD to CG; and because AE is equal to EC, the Square of AE will be equal to the Square of EC. But the Squares of EF, and FA, are † equal to the Square of AE; and the Squares of EG, and GC, equal † to the Square of EC. Therefore the Squares of EF, and FA, are equal to the Squares of EG, and GC. But the Square of EG is equal to the Square of EF; for EG is equal to EF. Therefore the Square of AF is equal to the Square of CG; and so AF is equal to CG. But AB is double to AF, and CD to CG. Therefore *equal Right Lines in a Circle are equally distant from the Centre; and Right Lines, which are equally distant from the Centre, are equal between themselves; which was to be demonstrated.*

PROPOSITION XV.

THEOREM.

A Diameter is the greatest Line in a Circle; and of all the other Lines therein, that which is nearest to the Centre is greater than that more remote.

LET ABCD be a Circle, whose Diameter is AD, and Centre E; and let BC be nearer to the Diameter than FG. I say, AD is the greatest, and BC is greater than FG.

For, let the Perpendiculars EH, EK, be drawn from the Centre E to BC, FG. Now, because BC is nearer to the Centre than FG, EK will be greater than EH. Let EL be equal to EH; draw LM through L at Right Angles to EK, which produce to N; and let EM, EN, EF, EG, be joined.

* 14 of this,

† 20. 1.

† 24. 1.

Then, because EH is equal to EL, the Line BC will be equal to MN*. And, since AE is equal to EM, and DE to EN, AD will be equal to ME and EN. But ME and EN are † greater than MN: And so AD is greater than MN; and NM is equal to BC. Therefore AD is greater than BC. And since the two Sides EM, EN, are equal to the two Sides FE, EG, and the Angle MEN greater than the Angle FEG, the Base MN shall be † greater than the Base FG. But MN is equal to BC. Therefore

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fore BC is greater than FG . And so the Diameter AD is the greatest, and BC is greater than FG . Wherefore the Diameter is the greatest Line in a Circle; and of all the other Lines therein, that which is nearest to the Centre is greater than that more remote; which was to be demonstrated.

PROPOSITION XVI.

THEOREM.

A Line drawn from the extreme [Point] of the Diameter of a Circle at Right Angles to that Diameter, shall fall without the Circle; and between the said Right Line, and the Circumference, no other Line can be drawn; and the Angle of a Semicircle is greater than any Right-lined acute Angle; and the remaining Angle [without any Circumference] is less than any Right-lined Angle.

LET ABC be a Circle, whose Centre is D , and Diameter AB . I say, a Right Line drawn from the Point A at Right Angles to AB , falls without the Circle.

For, if it does not, let it fall, if possible, within the Circle, as AC ; and join DC .

Now, because DA is equal to DC , the Angle DAC shall be * equal to the Angle ACD . But DAC is * 5. 1. a Right Angle; therefore ACD is a Right Angle: And accordingly the Angles DAC , ACD , are equal to two Right Angles; which is absurd †. Therefore † 17. 1. a Right Line drawn from the Point A at Right Angles to BA , will not fall within the Circle; and so likewise we prove, that it neither falls in the Circumference. Therefore it will necessarily fall without the same; which now let be AE .

Again, between the Right Line AE , and the Circumference CHA , no other Right Line can be drawn.

For, if there can, let it be FA , and let ‡ DG be ‡ 12. 1. drawn at Right Angles from the Centre D to FA .

Now, because AGD is a Right Angle, and DAG is less than a Right Angle, DA will be greater than DG *. But DA is equal to DH . Therefore DH is * 19. 1. greater

greater than DG , the less than the greater; which is absurd. Wherefore no right Line can be drawn between AE , and the Circumference AHC . I say, moreover, that the Angle of the Semicircle, contain'd under the right Line BA , and the Circumference CHA , is greater than any right-lin'd acute Angle; and the remaining Angle contain'd under the Circumference CHA , and the right Line AE , is less than any right-lin'd Angle.

For if any right-lin'd acute Angle be greater than the Angle contain'd under the right Line BA , and the Circumference CHA ; or if any right-lin'd Angle be less than that contain'd under the Circumference CHA , and the right Line AE , then a right Line may be drawn between the Circumference CHA and the right Line AE , making an Angle greater than that contain'd under the right Line BA , and the Circumference CHA , viz. which is contain'd under right Lines, and less than that contain'd under the Circumference CHA , and the right Line AE . But such a right Line cannot be drawn, from what has been prov'd. Therefore no right-lin'd acute Angle is greater than the Angle contain'd under the right Line BA , and the Circumference CHA ; nor less than the Angle contain'd under the Circumference CHA , and the right Line AE .

Coroll. From hence it is manifest, that a right Line drawn at right Angles on the End of the Diameter of a Circle, touches the Circle, and that in one Point only, because, if it should meet it in two Points, it would fall within the same; as has been demonstrated.

PROPOSITION XVII.

PROBLEM.

To draw a Right Line from a given Point, that shall touch a given Circle.

LET A be the Point given, and BCD the Circle. It is requir'd to draw a right Line from the Point A , that shall touch the given Circle BCD .

Let E be the Centre of the Circle, and join AE ; then about the Centre E , with the Distance EA , describe the Circle AFG ; draw DF * at right Angles * 11. 1. to EA , and join EBF , and AB . I say, the right Line AB is drawn from the Point A , touching the Circle BCD .

For, since E is the Centre of the Circles BCD , AFG , the Line EA will be equal to EF , and ED to EB . Therefore the two Sides AE , EB , are equal to the two Sides FE , ED , each to each; and they contain the common Angle E . Wherefore the Base DF is † equal to the Base AB , and the Triangle † 4. 1. DEF equal to the Triangle EBA , and the remaining Angles of the one equal to the remaining Angles of the other. And so the Angle EBA is equal to the Angle EDF . But EDF is a Right Angle. Wherefore EBA is also a Right Angle, and EB is a Line drawn from the Centre; but a right Line, drawn from the Extremity of the Diameter of a Circle at Right Angles † to it, touches the Circle. Wherefore † Cor. 16. of this AB touches the Circle; which was to be done.

PROPOSITION XVIII.

THEOREM

If any Right Line touches a Circle, and from the Centre to the Point of Contact a Right Line be drawn; that Line will be perpendicular to the Tangent.

LET any Right Line DE touch a Circle ABC in the Point C , and let there be drawn the Right Line FC from the Centre F . I say, FC is perpendicular to DE .

For, if it be not, let FG be drawn * from the * 12. 1. Centre F , perpendicular to DE .

Now, because the Angle FGC is a Right Angle, the Angle GCF will be † an acute Angle; and accordingly the Angle FGC is greater than the Angle FCG ; but the greater Side subtends † the greater † 19. 1. Angle. Therefore FC is greater than FG . But FC is equal to FB . Wherefore FB is greater than FG , a less than the greater; which is absurd. Therefore FG

FG is not perpendicular to DE. And in the same manner we prove, that no other Right Line but FC is perpendicular to DE. Wherefore FC is perpendicular to DE. Therefore, *if any Right Line touches a Circle, and from the Centre to the Point of Contact a Right Line be drawn, that Line will be perpendicular to the Tangent*; which was to be demonstrated.

PROPOSITION XIX.

THEOREM.

If any Right Line touches a Circle, and from the Point of Contact a Right Line be drawn at Right Angles to the Tangent, the Centre of the Circle shall be in the said Line.

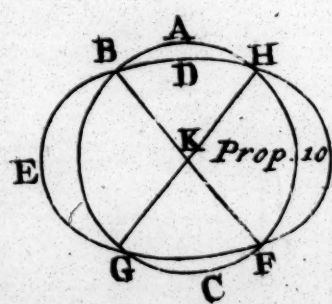
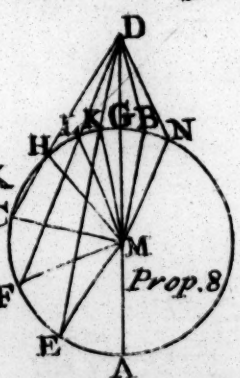
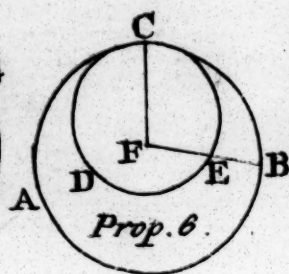
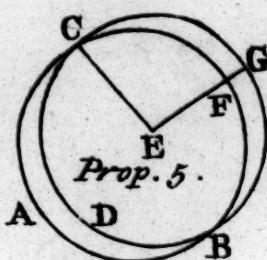
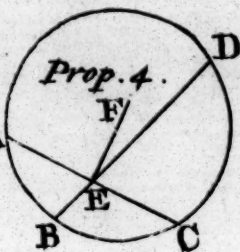
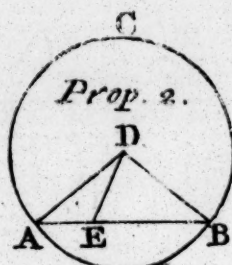
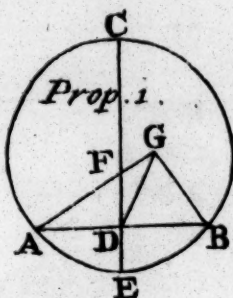
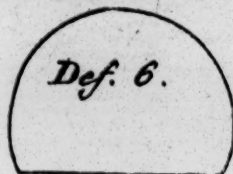
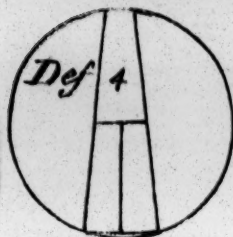
LET any Right Line DE touch the Circle ABC in C, and let CA be drawn from the Point C at Right Angles to DE. I say, the Circle's Centre is in AC.

For, if it be not, let F be the Centre, if possible; and join CF.

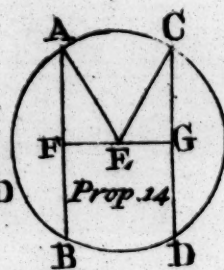
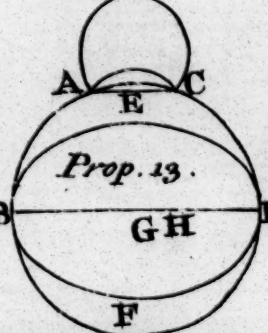
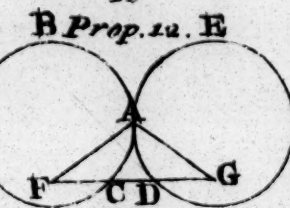
Then, because the Right Line DE touches the Circle ABC, and FC is drawn from the Centre to the Point of Contact; FC will be perpendicular to DE*. And so the Angle FCE is a right one. But ACE is also a right Angle†: Therefore the Angle FCE is equal to the Angle ACE, a less to a greater; which is absurd. Therefore F is not the Centre of the Circle ABC. After this manner we prove, that the Centre of the Circle can be in no other Line, unless in AC. Wherefore, *if any Right Line touches a Circle, and from the Point of Contact a Right Line be drawn at Right Angles to the Tangent, the Centre of the Circle shall be in the said Line*; which was to be demonstrated.

* 18 of this.

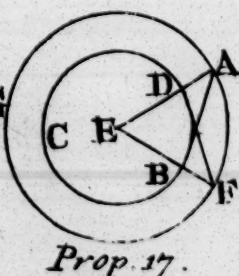
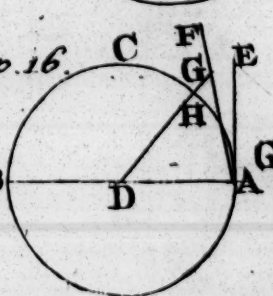
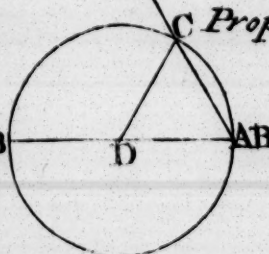
† From the Hyp.



Prop. 11.



Prop. 15.



FG is not perpendicular to DE . And in the same manner we prove, that no other Right Line but FC is perpendicular to DE . Wherefore FC is perpendicular to DE . Therefore, *if any Right Line touches a Circle, and from the Centre to the Point of Contact a Right Line be drawn, that Line will be perpendicular to the Tangent*; which was to be demonstrated.

PROPOSITION XIX.

THEOREM.

If any Right Line touches a Circle, and from the Point of Contact a Right Line be drawn at Right Angles to the Tangent, the Centre of the Circle shall be in the said Line.

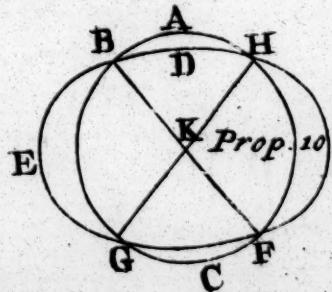
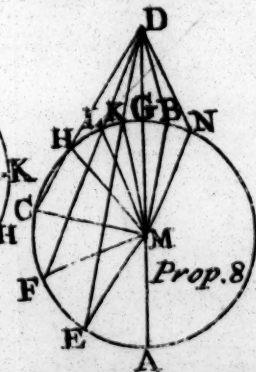
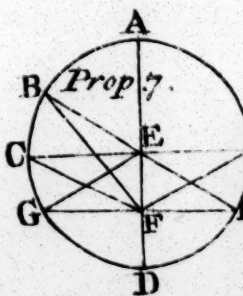
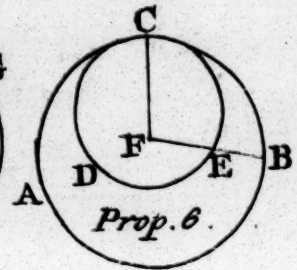
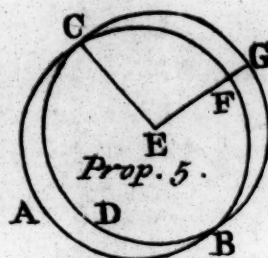
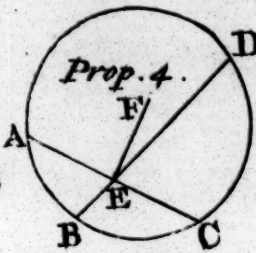
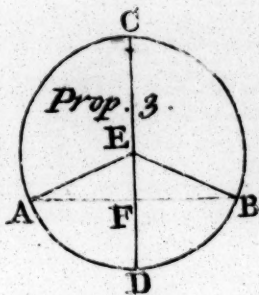
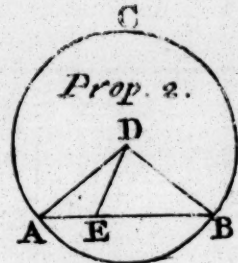
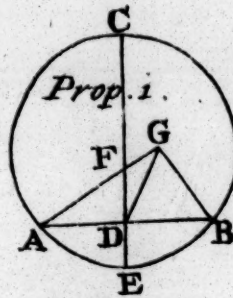
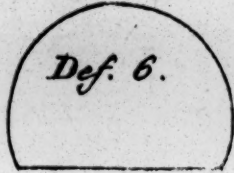
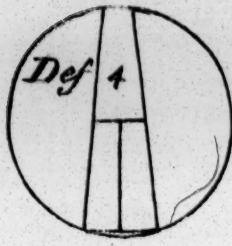
LET any Right Line DE touch the Circle ABC in C , and let CA be drawn from the Point C at Right Angles to DE . I say, the Circle's Centre is in AC .

For, if it be not, let F be the Centre, if possible; and join CF .

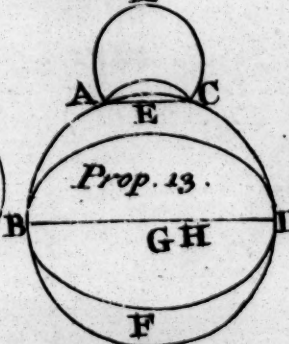
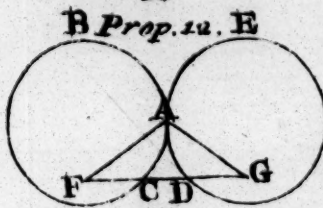
Then, because the Right Line DE touches the Circle ABC , and FC is drawn from the Centre to the Point of Contact; FC will be perpendicular to DE *. And so the Angle FCE is a right one. But ACE is also a right Angle†: Therefore the Angle FCE is equal to the Angle ACE , a less to a greater; which is absurd. Therefore F is not the Centre of the Circle ABC . After this manner we prove, that the Centre of the Circle can be in no other Line, unless in AC . Wherefore, *if any Right Line touches a Circle, and from the Point of Contact a Right Line be drawn at Right Angles to the Tangent, the Centre of the Circle shall be in the said Line*; which was to be demonstrated.

* 18 of this.

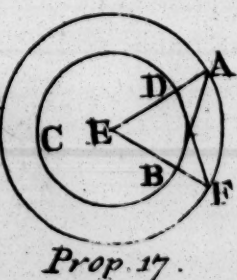
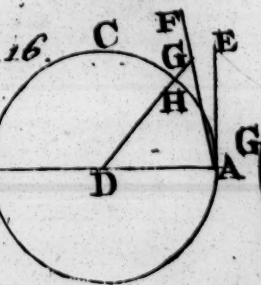
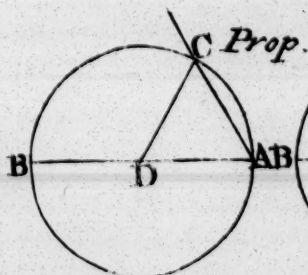
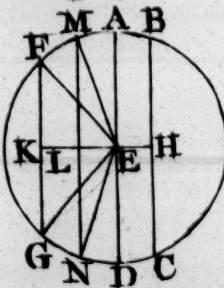
† From the Hyp.



Prop. 11.



Prop. 15.





PROPOSITION XX.

THEOREM.

The Angle at the Centre of a Circle is double to the Angle at the Circumference, when the same Arc is the Base of the Angles.

LET ABC be a Circle, at the Centre whereof is the Angle BEC, and at the Circumference the Angle BAC, both of which stand upon the same Arc BC. I say, the Angle BEC is double to the Angle BAC.

For join AE, and produce it to F.

Then, because EA is equal to EB, the Angle EAB shall be equal to the Angle EBA*. Therefore the * 5. 1. Angles EAB, EBA, are double to the Angle EAB; but the Angle BEF is † equal to the Angles EAB, † 32. 1. EBA; therefore the Angle BEF is double to the Angle EAB. For the same Reason, the Angle FEC is double to EAC. Therefore the whole Angle BEC is double to the whole Angle BAC. Again, let there be another Angle BDC; and join DE, which produce to G. We demonstrate, in the same manner, that the Angle GEC is double to the Angle GDC; whereof the Part GEB is double to the Part GDB. And therefore BEC is double to BDC. Consequently, *an Angle at the Centre of a Circle is double to the Angle at the Circumference, when the same Arc is the Base of the Angles; which was to be demonstrated.*

PROPOSITION XXI.

THEOREM.

Angles that are in the same Segment in a Circle, are equal to each other.

LET ABCDE be a Circle, and let BAD, BED, be Angles in the same Segment BAED. I say, those Angles are equal.

G

For,

For, let F be the Centre of the Circle $ABCDE$; and join BF , FD .

Now, because the Angle BFD is at the Centre, and the Angle BAD at the Circumference, and they stand upon the same Arc BCD ; the Angle BFD will be * double to the Angle BAD . For the same Reason, the Angle BFD is also double to the Angle BED . Therefore the Angle BAD will be equal to the Angle BED .

If the Angles BAD , BED , are in a Segment less than a Semicircle, let AE be drawn; and then all the Angles of the Triangle ABG are † equal to all the Angles of the Triangle DEG . But the Angles ABE , ADE , are equal, from what has been before prov'd; and the Angles AGB , DGE , are also equal ‡; for they are vertical Angles. Wherefore the remaining Angle BAG is equal to the remaining Angle GED . Therefore, *Angles that are in the same Segment in a Circle, are equal to each other*; which was to be demonstrated.

PROPOSITION XXII.

THEOREM.

The opposite Angles of any quadrilateral Figure, describ'd in a Circle, are equal to two right Angles.

LET $ABDC$ be a Circle, wherein is describ'd the quadrilateral Figure $ABCD$. I say, two opposite Angles thereof are equal to two right Angles.

For, join AD , BC .

Then, because the three Angles of any Triangle are * equal to two right Angles, the three Angles of the Triangle ABC , viz. the Angles CAB , ABC , BCA , are equal to two right Angles. But the Angle ABC is † equal to the Angle ADC ; for they are both in the same Segment $ABDC$. And the Angle ACB is ‡ equal to the Angle ADB , because they are in the same Segment $ACDB$; therefore the whole Angle BDC is equal to the Angles ABC , ACB ; and if the common Angle BAC be added, then the Angles BAC , ABC , ACB , are equal to the Angles BAC , BDC ; but the Angles BAC , ABC , ACB , are equal

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* equal to the two right Angles. Therefore likewise, * 32. 1.
the Angles BAC, BDC, shall be equal to two right Angles. And after the same way we prove; that the Angles ABD, ACD, are also equal to two right Angles. Therefore *the opposite Angles of any quadrilateral Figure, describ'd in a Circle, are equal to two right Angles*; which was to be demonstrated.

PROPOSITION XXIII.

THEOREM.

Two similar and unequal Segments of two Circles cannot be set upon the same right Line, and on the same Side thereof.

FOR if this be possible, let the two similar and unequal Segments ACB, ADB, of two Circles stand upon the right Line AB on the same Side thereof. Draw ACD, and let CB, BD, be join'd. Now, because the Segment ACB is similar to the Segment ADB, and similar Segments of Circles are * such * Def. 11. * which receive equal Angles; the Angle ACB will *this.* be equal to the Angle ADB, the outward one to the inward one; which is † absurd. Therefore *similar* † 16. 1. *and unequal Segments of two Circles cannot be set upon the same right Line, and on the same Side thereof*; which was to be demonstrated.

PROPOSITION XXIV.

THEOREM.

Similar Segments of Circles, being upon equal right Lines, are equal to one another.

LET AEB, CFD, be equal Segments of Circles standing upon the equal right Lines AB, CD. I say, the Segment AEB is equal to the Segment CFD. For the Segment AEB being applied to the Segment CFD, so that the Point A coincides with C, and the Line AB with CD; then the Point B will coincide with the Point D, since AB and CD are equal. And since the right Line AB coincides with CD,
G 2

CD, the Segment AEB will coincide with the Segment CFD. For if, at the same time that AB coincides with CD, the Segment AEB should not coincide with the Segment CFD, but be otherwise, as CGD; then a Circle would cut a Circle in more Points than two, viz. in the Points C, G, D; which is * impossible. Wherefore, if the right Line AB coincides with CD, the Segment AEB will coincide with, and be equal to, the Segment CFD. Therefore *similar Segments of Circles, being upon equal right Lines, are equal to one another*; which was to be demonstrated.

* 10 of this.

PROPOSITION XXV.

PROBLEM.

A Segment of a Circle being given, to describe the Circle whereof it is the Segment.

LET ABC be a Segment of a Circle given. It is requir'd to describe a Circle, whereof ABC is a Segment.

* 10. I.

† 11. I.

‡ 23. I.

* 6. I.

Bisect * AC in D, and let DB be drawn † from the Point D at right Angles to AC; and join AB. Now the Angle ABD is either greater, equal, or less than the Angle BAD. And first let it be greater, and make ‡ the Angle BAE at the given Point A, with the right Line BA, equal to the Angle ABD; produce DB to E, and join EC.

Then, because the Angle ABE is equal to the Angle BAE, the right Line BE will be * equal to EA. And because AD is equal to DC, and DE common, the two Sides AD, DE, are each equal to the two Sides CD, DE; and the Angle ADE is equal to the Angle CDE; for each is a right one. Therefore the Base AE is equal to the Base EC. But AE has been prov'd to be equal to EB. Wherefore BE is also equal to EC. And accordingly the three right Lines AE, EB, EC, are equal to each other. Therefore a Circle describ'd about the Centre E, with either of the Distances AE, EB, EC, shall pass thro' the other Points, and be that requir'd to be describ'd. But it is manifest, that the Segment ABC is less than a Semi-

a Semicircle, because the Centre thereof is without the same.

But if the Angle ABD be equal to the Angle BAD ; then if AD be made equal to BD, or DC, the three Right Lines AD, DB, DC, are equal between themselves, and D will be the Centre of the Circle to be describ'd ; and the Segment ABC is a Semicircle.

But if the Angle ABD is less than the Angle BAD, let the Angle BAE be made, at the given Point A with the right Line BA, within the Segment ABC, equal to the Angle ABD.

Then the Point E, in the right Line DB, will be the Centre, and ABC a Segment greater than a Semicircle. Therefore a Circle is describ'd, whereof a Segment is given ; *which was to be done.*

PROPOSITION XXVI.

THEOREM.

In equal Circles, equal Angles stand upon equal Circumferences, whether they be at their Centres, or at their Circumferences.

LET ABC, DEF, be equal Circles ; and let BGC, EHF, be equal Angles at their Centres ; and BAC, EDF, equal Angles at their Circumferences. I say, the Circumference BKC is equal to the Circumference ELF.

For, let BC, EF, be join'd. Because ABC, DEF, are equal Circles, the Lines drawn from their Centres will be equal. Therefore the two Sides BG, GC, are equal to the two Sides EH, HF ; and the Angle G is equal to the Angle at H. Wherefore the Base BC is * equal to the Base EF. Again, because the Angle at A is equal to that at D, the Segment BAC will be † similar to the Segment EDF ; and they are upon equal right Lines BC, EF. But those similar Segments of Circles that are upon equal right Lines, are † equal to each other. Therefore the Segment BAC will be † equal to the Segment EDF. But the whole Circle ABC is equal to the whole Circle DEF. Therefore the remaining Circumference BKC

G 3

shall

* 4. 1.

† Def. 11.

† 34 of this.

† Def. 11.

shall be equal to the remaining Circumference ELF . Therefore, in equal Circles, equal Angles stand upon equal Circumferences, whether they be at their Centres, or at their Circumferences; which was to be demonstrated.

PROPOSITION XXVII.

THEOREM.

Angles that stand upon equal Circumferences in equal Circles, are equal to each other, whether they be at their Centres or Circumferences.

LET the Angles BGC , EHF , at the Centres of the equal Circles ABC , DEF , and the Angles BAC , EDF , at their Circumferences, stand upon the equal Circumferences BC , EF . I say, the Angle BGC is equal to the Angle EHF , and the Angle BAC to the Angle EDF .

For if the Angle BGC be equal to the Angle EHF , it is manifest, that the Angle BAC is also equal to the Angle EDF : But if not, let one of them be the greater, as BGC , and make * the Angle BGK , at the Point G , with the Line BG , equal to the Angle EHF . But equal Angles stand † upon equal Circumferences, when they are at the Centres. Wherefore the Circumference BK is equal to the Circumference EF . But the Circumference EF is equal to the Circumference BC . Therefore BK is equal to BC , a less to a greater, which is absurd. Wherefore the Angle BGC is not unequal to the Angle EHF , and so it must be equal to it. But the Angle at A is one half of the Angle BGC ; and the Angle at D one half of the Angle EHF . Therefore the Angle at A is equal to the Angle at D . Wherefore *Angles that stand upon equal Circumferences in equal Circles, are equal to each other, whether they be at their Centres or Circumferences; which was to be demonstrated.*

* 23. I.

† 26 of this.

PROPOSITION XXVIII.

THEOREM.

In equal Circles, equal Right Lines cut off equal Parts of the Circumferences; the greater equal to the greater, and the lesser equal to the lesser.

LET ABC , DEF , be equal Circles, in which are the equal right Lines BC , EF , which cut off the greater Circumferences BAC , EDF , and the lesser Circumferences BGC , EHF . I say, the greater Circumference BAC is equal to the greater Circumference EDF , and the lesser Circumference BGC to the lesser Circumference EHF .

For, assume the Centres K and L of the Circles; and join BK , KC , EL , LF .

Because the Circles are equal, the Lines drawn from their Centres are * also equal. Therefore the * *Def. 1.* two Sides BK , KC , are equal to the two Sides EL , LF ; and the Base BC is equal to the Base EF . Therefore the Angle BKC is † equal to the Angle † *8. 1.* ELF . But equal Angles stand ‡ upon equal Circum- ‡ *26 of this.* ferences, when they are at the Centres. Wherefore the Circumference BGC is equal to the Circumference EHF , and the whole Circle ABC equal to the whole Circle DEF ; and so the remaining Circumference BAC shall be equal to the remaining Circumference EDF . Therefore, in equal Circles, equal right Lines cut off equal Parts of the Circumferences; *which was to be demonstrated.*

PROPOSITION XXIX.

THEOREM

In equal Circles, equal Right Lines subtend equal Circumferences.

LET there be two equal Circles, ABC , DEF ; and let the equal Circumferences BGC , EHF , be assum'd in them, and BC , EF , join'd. I say, the right Line BC is equal to the right Line EF .

G 4

For,

* 1 of this. For, find * the Centres of the Circles K, L; and join BK, KC, EL, LF.

Then, because the Circumference BGC is equal to the Circumference EHF, the Angle BKC shall be † equal to the Angle ELF. And because the Circles ABC, DEF, are equal, the Lines drawn from their Centres shall be † equal. Therefore the two Sides BK, KC, are equal to the two Sides EL, LF; and they contain equal Angles: Wherefore the Base BC is † equal to the Base EF. And so, in equal Circles, equal Circumferences subtend equal right Lines; which was to be demonstrated.

PROPOSITION XXX.

PROBLEM.

To cut a given Circumference into two equal Parts.

LET the given Circumference be ADB. It is requir'd to cut the same into two equal Parts.

* 10. I. Join AB, which biseet * in C; and let the right Line CD be drawn from the Point C at right Angles to AB; and join AD, DB.

Now, because AC is equal to CB, and CD is common, the two Sides AC, CD, are equal to the two Sides BC, CD; but the Angle ACD is equal to the Angle BCD; for each of them is a right Angle: Therefore the Base AD is † equal to the Base BD. But equal right Lines cut † off equal Circumferences. Wherefore the Circumference AD shall be equal to the Circumference BD. Therefore a given Circumference is cut into two equal Parts; which was to be done.

† 4. I.

† 28 of this.

P R O-

PROPOSITION XXXI.

THEOREM.

In a Circle, the Angle that is in a Semicircle, is a Right Angle; but the Angle in a greater Segment, is less than a Right Angle; and the Angle in a lesser Segment, greater than a Right Angle: Moreover, the Angle of a greater Segment is greater than a Right Angle; and the Angle of a lesser Segment is less than a Right Line.

LET there be a Circle $ABCD$, whose Diameter is BC , and Centre E ; and join BA , AC , AD , DC . I say, the Angle which is in the Semicircle BAC is a Right Angle; that which is in the Segment ABC being greater than a Semicircle, viz. the Angle ABC , is less than a Right Angle; and that which is in the Segment ADC , being less than a Semicircle, that is, the Angle ADC , is greater than a Right Angle.

For, join AE , and produce BA to F .

Then, because BE is equal to EA , the Angle EAB shall be * equal to the Angle EBA . And because * 5. 1. AE is equal to EC , the Angle ACE will be * equal to the Angle CAE . Therefore the whole Angle BAC is equal to the two Angles ABC , ACB ; but the Angle FAC , being without the Triangle ABC , is † equal to the two Angles ABC , ACB : † 32. 1. Therefore the Angle BAC is equal to the Angle FAC ; and so each of them is † a Right Angle. † Def. 10. 1. Wherefore the Angle BAC in a Semicircle is a Right Angle. And because the two Angles ABC , BAC , of the Triangle ABC *, are less than two Right * 17. 1. Angles, and BAC is a Right Angle; then ABC is less than a Right Angle, and is in the Segment ABC greater than a Semicircle.

And since $ABCD$ is a quadrilateral Figure in a Circle, and the opposite Angles of any quadrilateral Figure described in a Circle, are † equal to two Right † 12 of this. Angles; the Angles ABC , ADC , are equal to two Right Angles, and the Angle ABC is less than a Right Angle: Therefore the remaining Angle ADC will

will be greater than a Right Angle, and is in the Segment ADC , which is less than a Semicircle.

I say, moreover, the Angle of the greater Segment contained under the Circumference ABC , and the Right Line AC , is greater than a Right Angle; and the Angle of the lesser Segment, contained under the Circumference ADC , and the Right Line AC , is less than a Right Angle. This manifestly appears; for, because the Angle contained under the Right Lines BA , AC , is a Right Angle, the Angle contained under the Circumference ABC , and the Right Line AC , will be greater than a Right Angle. Again, because the Angle contained under the Right Line CA , AF , is a Right Angle, therefore the Angle which is contained under the Right Line AC , and the Circumference ADC , is less than a Right Angle. Therefore, *in a Circle, the Angle that is in a Semicircle, is a Right Angle; but the Angle in a greater Segment is less than a Right Angle; and the Angle in a lesser Segment greater than a Right Angle: Moreover, the Angle of a greater Segment is greater than a Right Angle; and the Angle of a lesser Segment is less than a Right Angle; which was to be demonstrated.*

PROPOSITION XXXII.

THEOREM.

If any Right Line touches a Circle, and a Right Line be drawn from the Point of Contact cutting the Circle; the Angles it makes with the Tangent Line, will be equal to those which are made in the alternate Segments of the Circle.

LET any Right Line EF touch the Circle $ABCD$ in the Point B , and let the Right Line BD be any-how drawn from the Point B cutting the Circle. I say, the Angles which BD makes with the Tangent Line EF , are equal to those in the alternate Segments of the Circle; that is, the Angle FBD is equal to an Angle made in the Segment DAB , viz. to the Angle DAB ; and the Angle DBE equal to the Angle DCB , made in the Segment DCB . For,

Draw

Draw * BA from the Point B at Right Angles to * 11. 1.
EF; and take any Point C in the Circumference
BD; and join AD, DC, CB.

Then, because the Right Line EF touches the Circle ABCD in the Point B; and the Right Line BA is drawn from the Point of Contact B at Right Angles to the Tangent Line; the Centre of the Circle ABCD will † be in the Right Line BA; and so † 19. 3.
BA is a Diameter of the Circle, and the Angle ADB, in a Semicircle, is † a Right Angle. Therefore the † 31 of this.
other Angles BAD, ABD, are * equal to one Right * 32. 1.
Angle. But the Angle ABF is also a Right Angle: Therefore the Angle ABF is equal to the Angles BAD, ABD; and if ABD, which is common, be taken away, then the Angle DBF remaining, will be equal to that which is in the alternate Segment of the Circle, viz. equal to the Angle BAD. And because ABCD is a quadrilateral Figure in a Circle, and the opposite Angles thereof are † equal to two † 22 of this.
Right Angles; the Angles DBF, DBE, will be equal to the Angles BAD, BCD. But BAD has been proved to be equal to DBF; therefore the Angle DBE is equal to the Angle made in DCB, the alternate Segment of the Circle, viz. equal to the Angle DCB. Therefore, *if any Right Line touches a Circle, and a Right Line be drawn from the Point of Contact cutting the Circle; the Angles it makes with the Tangent Line, will be equal to those which are made in the alternate Segments of the Circle; which was to be demonstrated.*

PROPOSITION XXXIII.

PROBLEM.

To describe, upon a given Right Line, a Segment of a Circle, which shall contain an Angle equal to a given Right-lined Angle.

LET the given Right Line be AB, and C the given Right-lined Angle. It is required to describe the Segment of a Circle upon the given Right Line AB, containing an Angle equal to the Angle C.

At

At the Point A, with the Right Line A B, make
 † 23. I. † the Angle B A D equal to the Angle C, and draw
 * 11. I. * A E from the Point A, at Right Angles, to A D.
 † 10. I. Likewise bisect † A B in F, and let F G be drawn
 from the Point F, at Right Angles, to A B, and
 join G B.

Then, because A F is equal to F B, and F G is com-
 mou, the two Sides A F, F G, are equal to the two
 Sides B F, F G; and the Angle A F G is equal to the
 † 4. I. Angle B F G. Therefore the Base A G is † equal
 to the Base G B. And so, if a Circle be described
 about the Centre G, with the Distance A G, this shall
 pass through the Point B. Describe the Circle, which
 let be A B E, and join E B. Now, because A D is
 drawn from the Point A, the Extremity of the Diame-
 * Cor. 16. ter A E, at the Right Angles to A E, the said A D will *
 of this. touch the Circle. And since the Right Line A D
 touches the Circle A B E, and the Right Line A B is
 drawn in the Circle from the Point of Contact A,
 † 32 of this. the Angle D A B is † equal to the Angle made in the
 alternate Segment, viz. equal to the Angle A E B.
 But the Angle D A B is equal to the Angle C. There-
 fore the Angle C will be equal to the Angle A E B.
 Wherefore the Segment of a Circle A E B is described
 upou the given Right Line A B, containing an Angle
 A E B, equal to a given Angle C; *which was to be done.*

PROPOSITION XXXIV.

THEOREM.

*To cut off a Segment from a given Circle, that
 shall contain an Angle equal to a given Right-
 lin'd Angle.*

L E T the given Circle be A B C, and the Right-
 lined Angle given D. It is required to cut off a
 Segment from the Circle A B C, containing an An-
 gle equal to the Angle D.

† 17 of this. Draw † the Right Line E F, touching the Circle in
 * 23. I. the Point B, and make * the Angle F B C at the Point
 B equal to the Angle D.

Then, because the Right Line E F touches the Cir-
 cle A B C in the Point B, and B C is drawn from
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the Point of Contact B; the Angle FBC will be * equal * ^{32 of this.} to that in the alternate Segment of the Circle; but the Angle FBC is equal to the Angle D. Therefore the Angle in the Segment BAC will be equal to the Angle D. Therefore the Segment BAC is cut off from the given Circle ABC, containing an Angle equal to the given Right-lined Angle D; which was to be done.

PROPOSITION XXXV.

THEOREM.

If two Right Lines in a Circle mutually cut each other, the Rectangle contained under the Segments of the one, is equal to the Rectangle under the Segments of the other.

IN the Circle ABCD, let two Right Lines mutually cut each other in the Point E. I say, the Rectangle contained under AE, and EC, is equal to the Rectangle contained under DE, EB.

If AC and DB pass through the Centre, so that E be the Centre of the Circle ABCD; it is manifest, since AE, EC, DE, EB, are equal; that the Rectangle under AE, EC, is equal to the Rectangle under DE, EB.

But if AC, DB, do not pass through the Centre, assume the Centre of the Circle F; from which draw FG, FH, perpendicular to the Right Lines AC, DB; and join FB, FC, FE.

Then, because the Right Line GF, drawn through the Centre, cuts the Right Line AC, not drawn through the Centre, at Right Angles, it will also bisect * the * ^{4 of this.} same. Wherefore AG is equal to GC: And because the Right Line AC is cut into two equal Parts in the Point G, and into two unequal Parts in E, the Rectangle under AE, EC, together with the Square of EG, is † equal to the Square of GC. And if † ^{5. 2.} the common Square of GF be added, then the Rectangle under AE, EC, together with the Squares of EG, GF, is equal to the Squares of CG, GF. But the Square of FE is † equal to the Squares of EG, † ^{47. 1.} GF, and the Square of FC equal † to the Squares of

of CG, GF . Therefore the Rectangle under AE, EC , together with the Square of FE , is equal to the Square of FC , but CF is equal to FB . Therefore the Rectangle under AE, EC , together with the Square of FE , is equal to the Square of FB . For the same Reason, the Rectangle under DE, EB , together with the Square of FE , is equal to the Square of FB . But it has been proved, that the Rectangle under AE, EC , together with the Square of FE , is also equal to the Square of FB . Therefore the Rectangle under AE, EC , together with the Square of FE , is equal to the Rectangle under DE, EB , together with the Square of FE . And if the common Square of FE be taken away, then there will remain the Rectangle under AE, EC , equal to the Rectangle under DE, EB . Wherefore, *if two Right Lines in a Circle mutually cut each other, the Rectangle contained under the Segments of the one, is equal to the Rectangle under the Segments of the other; which was to be demonstrated.*

PROPOSITION XXXVI.

THEOREM.

If some Point be taken without a Circle, and from that Point two Right Lines fall to the Circle, one of which cuts the Circle, and the other touches it; the Rectangle contained under the whole Secant Line, and its Part between the Convexity of the Circle and the assumed Point, will be equal to the Square of the Tangent Line.

LET any Point D be assumed without the Circle ABC , and let two Right Lines DCA, DB , fall from the said Point to the Circle; whereof DCA cuts the Circle, and DB touches it. I say, the Rectangle under AD, DC , is equal to the Square of DB .

Now, DCA either passes thro' the Centre, or not. In the first Place, let it pass thro' the Centre of the Circle ABC , which let be E , and join EB . Then
 * is of this. the Angle EBD is * a Right Angle. And so, since the Right Line AC is bisected in E , and CD is added thereto, the Rectangle under AD, DC , together with

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with the Square of EC , shall * be equal to the Square * 6. 2.
 of ED . But EC is equal to EB ; wherefore the
 Rectangle under AD , DC , together with the Square
 of EB , is equal to the Square of ED . But the Square
 of ED is † equal to the Square of EB , and BD . For † 47. 1.
 the Angle EBD is a Right Angle: Therefore the
 Rectangle under AD , DC , together with the Square
 of EB , is equal to the Squares of EB and BD ; and
 if the common Square of EB be taken away, the
 Rectangle under AD , DC , remaining, will be equal
 to the Square of the Tangent Line BD .

Now, let DCA not pass through the Centre of the
 Circle ABC ; and find † the Centre E thereof, and † 1 of this.
 draw EF perpendicular to AC , and join EB , EC ,
 ED . Therefore EFD is a Right Angle. And be-
 cause a Right Line EF , drawn through the Centre,
 cuts a Right Line AC at Right Angles, not drawn
 through the Centre, it will * bisect the same at Right * 3 of this.
 Angles; and so AF is equal to FC . Again, since
 the Right Line AC is bisected in F , and CD is added
 hereto, the Rectangle under AD , DC , together
 with the Square of FC , will be * equal to the Square
 of FD . And if the common Square of EF be added,
 then the Rectangle under AD , DC , together with
 the Squares of FC and FE , is equal to the Squares
 of DF and FE . But the Square of DE is equal
 to the Squares of DF and FE ; for the Angle EFD is
 Right one: And the Square of CE is † equal to the
 Square of CF and FE . Therefore the Rectangle
 under AD , DC , together with the Square of CE ,
 is equal to the Square of ED ; but CE is equal to
 EB . Wherefore the Rectangle under AD , DC , to-
 gether with the Square of EB , is equal to the Square
 of ED . But the Squares of EB and BD are † equal
 to the Square of ED ; since the Angle EBD is a Right
 one. Wherefore the Rectangle under AD and DC ,
 together with the Square of EB , is equal to the
 Squares of EB and BD . And if the common Square
 of EB be taken away, the Rectangle under AD and
 DC , remaining, will be equal to the Square of DB .
 Therefore, if any Point be taken without a Circle, and
 from that Point two Right Lines fall to the Circle, one
 of which cuts the Circle, and the other touches it; the
 Rectangle contained under the whole Secant Line, and
 its

its Part between the Convexity of the Circle and the assumed Point, will be equal to the Square of the Tangent Line; which was to be demonstrated.

PROPOSITION XXXVII.

THEOREM.

If some Point be taken without a Circle, and two Right Lines be drawn from it to the Circle, so that one cuts it, and the other falls upon it; and if the Rectangle under the whole Secant Line, and the Part thereof, without the Circle, be equal to the Square of the Line falling upon the Circle, then this last Line will touch the Circle.

LET some Point D be assumed without the Circle ABC, and from it draw two Right Lines DCA, DB, to the Circle, in such manner, that DCA cuts the Circle, and DB falls upon it: And let the Rectangle under AD, DC, be equal to the Square of DB. I say, the Right Line DB touches the Circle.

* 17 of this. For, let the Right Line DE be drawn * touching the Circle ABC, and find F the Centre of the Circle, and join EF, FB, FD.

† 18 of this. Then, the Angle FED is † a Right Angle. And because DE touches the Circle ABC, and DCA cuts it, the Rectangle under AD, and DC, will be equal to the Square of DE. But the Rectangle under AD and DC is ‡ equal to the Square of DB. Wherefore the Square of DE shall be equal to the Square of DB. And so the Line DE will be equal to the Line DB. But EF is equal to FB: Therefore the two Sides DE, EF, are equal to the two Sides DB, BF; and the Base FD is common. Wherefore the Angle DEF is equal to the Angle DBF; but DEF is a Right Angle; wherefore DBF is also a Right Angle, and FB produced is a Diameter. But a Right Line drawn at Right Angles, on the End of the Diameter of a Circle, touches the Circle; therefore BD necessarily touches the Circle. We prove this in the same manner, if the Centre of the Circle be in the Right Line CA. If therefore any Point be assumed without a Circle, and two Right Lines

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Lines be drawn from it to the Circle, so that one cuts, and the other falls upon it; and if the Rectangle under the whole Secant Line, and the Part thereof, without the Circle, be equal to the Square of the Line falling upon the Circle; then this last Line will touch the Circle; which was to be demonstrated.

Propoll. Hence, if from any Point, without a Circle, several right Lines AB , AC , are drawn, cutting the Circle; the Rectangles comprehended under the whole Lines AB , AC , and their external Parts AE , AF , are equal between themselves. For, if the Tangent AD be drawn, the Rectangle under BA , and AE , is equal to the Square of AD ; and the Rectangle under CA , and AF , is equal to the same Square of AD : Therefore the Rectangles shall be equal.

The END of the THIRD BOOK.

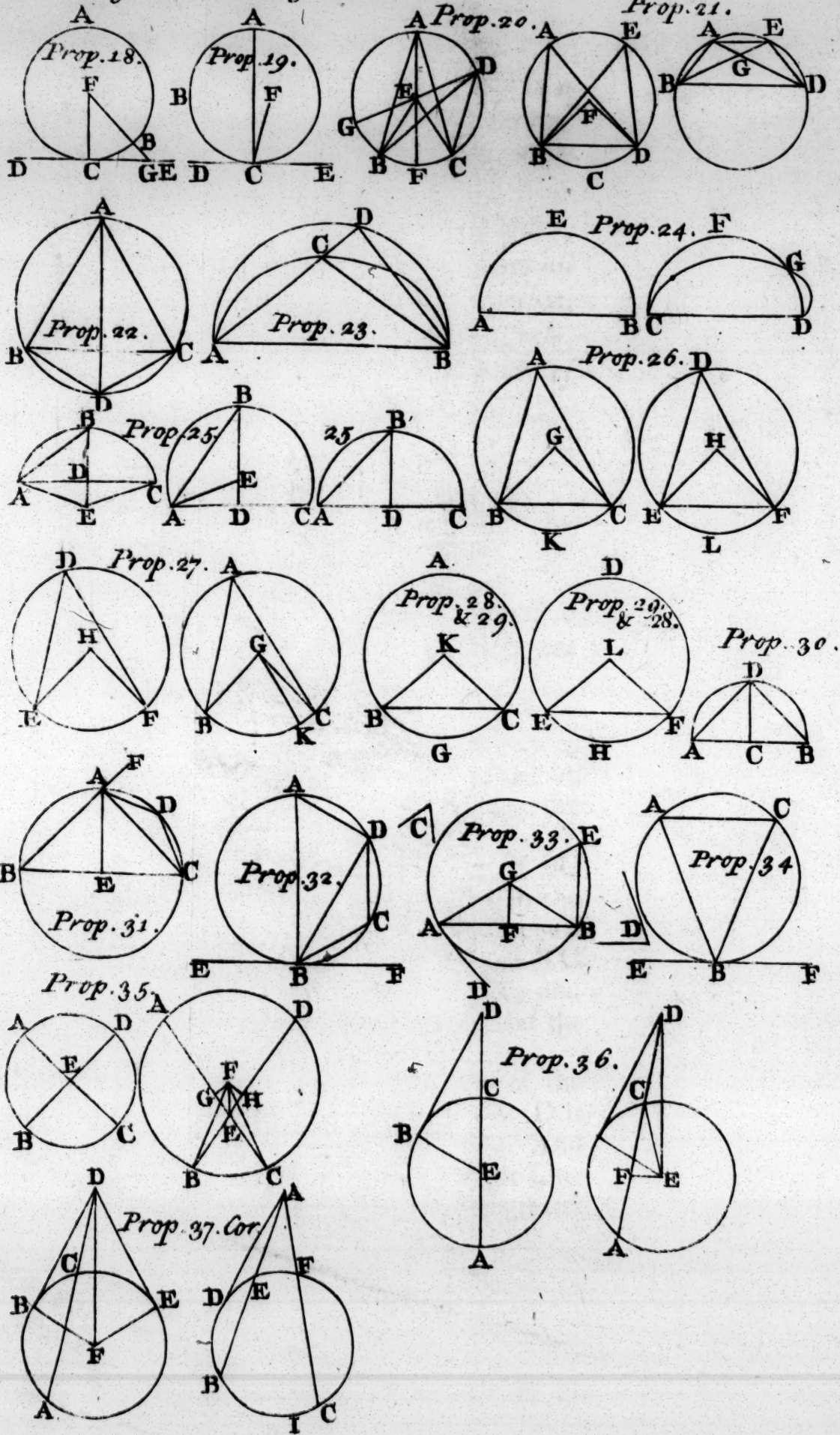
E U C L I D's ELEMENTS

B O O K IV.

DEFINITIONS.

- I. **A** Right-lined Figure is said to be inscribed in a right-lined Figure, when every one of the Angles of the inscribed Figure touches every one of the Sides of the Figure, wherein it is described.
- II. In like manner a Figure is said to be described about a Figure, when every one of the Sides of the Figure, circumscribed, touches every one of the Angles of the Figure, about which it is circumscribed.
- III. A right-lined Figure is said to be inscribed in a Circle, when every one of the Angles of that Figure which is inscribed, touches the Circumference of the Circle.
- IV. A right-lined Figure is said to be described about a Circle, when every one of the Sides of the circumscribed Figure touches the Circumference of the Circle.

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- V. *So likewise a Circle is said to be inscribed in a right-lined Figure, when the Circumference of the Circle touches all the Sides of the Figure in which it is inscribed.*
- VI. *A Circle is said to be described about a Figure, when the Circumference of the Circle touches all the Angles of the Figure which it circumscribes.*
- VII. *A right Line is said to be applied in a Circle, when its Extremes are in the Circumference of the Circle.*

PROPOSITION I.

PROBLEM.

To apply a Right Line in a given Circle, equal to a given Right Line, whose Length does not exceed the Diameter of the Circle.

LET the Circle given be ABC , and the given right Line, not greater than the Diameter, be D . It is requir'd to apply a right Line in the Circle ABC , equal to the right Line D .

Draw BC the Diameter of the Circle; then, if BC be equal to D , what was requir'd is done: For in the Circle ABC there is applied the right Line BC , equal to the right Line D : But if not, the Diameter BC is greater than D , and put $* CE$ equal to D ; and $* 3. 1.$ about the Centre C , with the Distance CE , let the Circle AEF be describ'd; and join CA .

Then, because the Point C is the Centre of the Circle AEF , CA will be equal to CE ; but D is equal to CE . Wherefore AC is equal to D . And so, in the Circle ABC , there is applied a right Line AC , equal to the given right Line D , not greater than the Diameter; *which was to be done.*

PROPOSITION II.

PROBLEM.

In a given Circle to describe a Triangle equiangular to a given Triangle.

LET ABC be a Circle given, and DEF a given Triangle. It is requir'd to describe a Triangle in the Circle ABC , equiangular to the Triangle DEF .
 * 17. 3. Draw the right Line GAH touching * the Circle ABC in the Point A , and with the right Angle AH ,
 † 23. 1. at the Point A , make † an Angle HAC , equal to the Angle DEF . Likewise, at the same Point A , with the Line AG , make the Angle GAB equal to the Angle DFE ; and join BC .

Then, because the right Line HAG touches the Circle ABC , and AC is drawn from the Point of Contact in the Circle,, the Angle HAC shall be † equal to ABC , the Angle in the alternate Segment of the Circle. But the Angle HAC is equal to the Angle DEF ; therefore also the Angle ABC is equal to the Angle DEF : For the same Reason, the Angle ACB is likewise equal to the Angle DFE . Wherefore the other Angle BAC shall be † equal to the other Angle EDF ; and, consequently, the Triangle ABC is equiangular to the Triangle DEF , and is describ'd in the Circle ABC ; *which was to be done.*

PROPOSITION III.

PROBLEM.

About a given Circle to describe a Triangle, equiangular to a Triangle given.

LET ABC be the given Circle, and DEF the given Triangle. It is requir'd to describe a Triangle about the Circle ABC , equiangular to the Triangle DEF .

Produce the Side EF both ways to the Points G and H , and find the Centre of the Circle K , and anyhow draw the Line KB . Then at the Point K , with KB

KB make * the Angle BKA equal to the Angle * 23. 1.
DEG; and the Angle BKC at the same Point K
on the other Side the Line KB, equal to the Angle
DFH; and thro' the Points A, B, C, let the right
Lines LAM, MBN, NCL, be drawn touching
the Circle ABC.

Then, because the Lines LM, MN, NL, touch
the Circle ABC in the Points A, B, C, and the Lines
KA, KB, KC, are drawn from the Centre K to the
Points A, B, C; the Angles at the Points A, B, C,
will be † right Angles. And because the four An- † 18. 3.
gles of the quadrilateral Figure AMBK are equal to
four right Angles, (for it may be divided into two
Triangles) and the Angles KAM, KBM, are each
right Angles; therefore the other Angles AKB,
AMB, are equal to two right Angles. But DEG,
DEF, are equal to two right Angles; therefore the
Angles AKB, AMB, are equal to the Angles DEG,
DEF, whereof AKB is equal to DEG. Where-
fore the other Angle AMB is equal to the other An-
gle DEF. In like manner we demonstrate, that the
Angle LNB is equal to the Angle DFE. Therefore
the other Angle MLN is † equal to the other Angle † Cor. 2.
EDF. Wherefore the Triangle LNM is equian- 32. 1.
gular to the Triangle DEF, and is describ'd about
the Circle ABC; which was to be done.

PROPOSITION IV.

PROBLEM.

To inscribe a Circle in a given Triangle.

LET ABC be a Triangle given. It is requir'd to
inscribe a Circle in the same.

Cut * the Angles ABC, BCA, into two equal * 9. 1.
Parts by the right Lines BD, DC, meeting each
other in the Point D. And from this Point draw
DE, DF, DG, † perpendicular to the Sides AB, † 12. 1.
BC, AC.

Now, because the Angle EBD is equal to the An-
gle FBD, and the right Angle BED is equal to the
right Angle BFD; then the two Triangles EBD,
DBF, have two Angles of the one equal to two
Angles

* 26. 1.

† 16. 3.

Angles of the other, and one Side DB common to both, viz. that which subtends the equal Angles; therefore the other Sides of the one Triangle shall be * equal to the other Sides of the other; and so DE shall be equal to DF. And for the same Reason, DG is equal to DF. Therefore DE is also equal to DG. And so the three right Lines DE, DF, DG, are equal between themselves. Wherefore a Circle describ'd about the Centre D, with either of the Distances DE, DF, DG, will also pass thro' the other Points. And the Sides AB, BC, AC, will touch it; because the Angles at E, F, and G, are right Angles. For if it should cut them, a right Line drawn on the Extremity of the Diameter of a Circle at right Angles, will fall without the Circle; which is † absurd. Therefore a Circle describ'd about the Centre D, with either of the Distances DE, DF, DG, will not cut the Sides AB, BC, CA; wherefore it will touch them, and will be a Circle describ'd in the Triangle ABC. Therefore the Circle EFG is describ'd in the given Triangle ABC; which was to be done.

PROPOSITION V.

PROBLEM.

To describe a Circle about a given Triangle.

LET ABC be a given Triangle. It is requir'd to describe a Circle about the same.

* 10. 1.

† 11. 1.

Bisect * the Sides AB, AC, in the Points D, E, from which Points let DF, EF, be drawn † at right Angles to AB, AC, which will meet either within the Triangle ABC, or in the Side BC, or without the Triangle.

‡ 4. 1.

First, Let them meet in the Point F within the Triangle, and join BF, FC, FA. Then, because AD is equal to DB, and DF is common, and at right Angles to AB; the Base AF will be ‡ equal to the Base FB. And after the same manner we prove, that the Base CF is equal to the Base FA. Therefore also BF equal to CF: And so the three right Lines FA, FB, FC, are equal to each other. Wherefore a Circle describ'd about the Centre F, with either of the Distances

Distances FA, FB, FC, will pass also thro' the other Points, and will be a Circle describ'd about the Triangle ABC. Therefore describe the Circle ABC.

Secondly, Let DF, EF, meet each other in the Point F, in the Side BC, as in the Second Figure, and join AF. Then we prove, as before, that the Point F is the Centre of a Circle describ'd about the Triangle ABC.

Lastly, Let the right Lines DF, EF, meet one another again in the Point F, without the Triangle, as in the third Figure, and join AF, FB, FC. And because AD is equal to DB, and DF is common, and at right Angles, the Base AF shall be equal to the Base BF. So likewise we prove, that CF is also equal to AF. Wherefore BF is equal to CF. And so again, if a Circle be describ'd on the Centre F, with either of the Distances FA, FB, FC, it will pass through the other Points, and will be describ'd about the Triangle ABC; *which was to be done.*

Coroll. If a Triangle be right-angled, the Centre of the Circle falls in the Side opposite to the right Angle; if acute-angled, it falls within the Triangle; and if obtuse-angled, it falls without the Triangle.

PROPOSITION VI.

PROBLEM.

To inscribe a Square in a given Circle.

LET ABCD be a Circle given. It is requir'd to inscribe a Square within the same.

Draw AC, DB, two Diameters of the Circle, cutting one another at right Angles; and join AB, BC, CD, DA.

Then, because BE is equal to ED, (for E is the Centre) and EA is common, and at right Angles to BD, the Base BA shall be * equal to the Base AD; * 4. 1. and for the same Reason BC, CD, as also BA, AD, are all equal to each other. Therefore the quadrilateral Figure ABCD is equilateral. I say, it is also rectangular. For, because the right Line DB is a Diameter of the Circle ABCD, BAD will be a Se-

* 31. 3.

micircle. Wherefore the Angle BAD is * a right Angle. And for the same Reason every one of the Angles ABC , BCD , CDA , is a right Angle. Therefore $ABCD$ is a rectangular quadrilateral Figure; But it has also been prov'd to be equilateral. Wherefore it shall necessarily be a Square, and is describ'd in the Circle $ABCD$; *which was to be done.*

PROPOSITION VII.

PROBLEM.

To describe a Square about a given Circle.

LET $ABCD$ be a Circle given. It is requir'd to describe a Square about the same.

* 17. 3.

Draw AC , BD , two Diameters of the Circle cutting each other at right Angles; and through the Points A , B , C , D , draw * FG , GH , HK , KF , Tangents to the Circle $ABCD$.

† 18. 3.

‡ 28. 1.

Then, because FG touches the Circle $ABCD$, and EA is drawn from the Centre E to the Point of Contact A , the Angles at A will be † right Angles. For the same Reason, the Angles at the Points B , C , D , are right Angles. And since the Angle AEB is a right Angle, as also EBG , GH shall be ‡ parallel to AC ; and for the same Reason, AC to KF . In this manner we prove likewise, that GF and HK are parallel to BD ; and so GF is parallel to HK . Therefore GK , GC , AK , FB , BK , are Parallelograms; and so GF is * equal to HK , and GH to FK . And since AC is equal to BD , and AC * equal to either GH , or FK ; and BD equal to either GF , or HK ; GH , or FK , is equal to GF , or HK . Therefore $FGHK$ is an equilateral quadrilateral Figure: I say it is also equiangular. For, because $GBEA$ is a Parallelogram, and AEB is a right Angle, then AGB shall be also a right Angle. In like manner we demonstrate, that the Angles at the Points H , K , F , are right Angles. Therefore the quadrilateral Figure $FGHK$ is rectangular; but it has been prov'd to be equilateral likewise. Wherefore it must necessarily be a Square, and is describ'd about the Circle $ABCD$; *which was to be done.*

* 34. 1.

PRO-

PROPOSITION VIII.

PROBLEM.

To describe a Circle in a given Square.

LET the given Square be $ABCD$. It is requir'd to describe a Circle within the same.

Bisect * the Sides AB , AD , in the Points F , E ; * 10. 1. and draw $\dagger$ EH thro' E , parallel to AB , or DC ; and $\dagger$ 31. 1. FK thro' F , parallel $\dagger$ to BC , or AD . Then AK , KB , AH , HD , AG , GC , BG , GD , are all Parallelograms, and their opposite Sides are $\dagger$ equal. And be- $\dagger$ 34. 1. cause DA is equal to AB , and AE is half of AD , and AF half of AB , AE shall be equal to AF ; but the opposite Sides are also equal. Therefore FG is equal to GE . In like manner we demonstrate, that GH , or GK , is equal to either FG , or GE . Therefore GE , GF , GH , GK , are equal to each other: And so a Circle being describ'd about the Centre G , with either of the Distances GE , GF , GH , GK , will also pass thro' the other Points, and shall touch the Sides AB , BC , CD , DA , because the Angles at E , F , H , K , are right Angles. For if the Circle should cut the Sides of the Square, a right Line, drawn from the End of the Diameter of a Circle, at right Angles, will fall within the Circle; which is * absurd. Wherefore a Circle describ'd about the * 16. 3. Centre G , with either of the Distances GE , GF , GH , GK , will not cut AB , BC , CD , DA , the Sides of the Square. Wherefore it shall necessarily touch them, and will be describ'd in the Square $ABCD$; which was to be done.

PROPOSITION IX.

PROBLEM.

To describe a Circle about a Square given.

LET $ABCD$ be a Square given. It is requir'd to circumscribe a Circle about the same.

Join AC , BD , mutually cutting one another in the Point E . And

* 3. 1.

And since DA is equal to AB , and AC is common, the two Sides DA, AC , are equal to the two Sides BA, AC ; but the Base DC is equal to the Base BC . Therefore the Angle DAC will * be equal to the Angle BAC : And consequently the Angle DAB is bisected by the right Line AC . In the same manner we prove, that each of the Angles ABC, BCD, CDA , are bisected by the right Lines AC, DB .

† 6. 1.

Then, because the Angle DAB is equal to the Angle ABC , and the Angle EAB is half of the Angle DAB , and the Angle EBA half of the Angle ABC ; the Angle EAB shall be equal to the Angle EBA : And so the Side EA is † equal to the Side EB . In like manner we demonstrate, that each of the right Lines, EC, ED , is equal to each of the right Lines EA, EB . Therefore the four right Lines EA, EB, EC, ED , are equal between themselves. Wherefore a Circle being describ'd about the Centre E , with either of the Distances EA, EB, EC, ED , will also pass thro' the other Points, and will be describ'd about the Square $ABCD$; which was to be done.

PROPOSITION X.

PROBLEM.

To make an Isosceles Triangle, having each of the Angles at the Base double to the other Angle.

* 11. 2.

† 1 of this.

‡ 5 of this.

CUT * any given right Line AB in the Point C , so that the Rectangle contain'd under AB, BC , be equal to the Square of AC ; then about the Centre A , with the Distance AB , let the Circle BDE be describ'd; and † in the Circle BDE apply the right Line BD equal to AC ; which is not greater than the Diameter. This being done, join DA, DC , and describe ‡ a Circle ACD about the Triangle ADC .

Then, because the Rectangle ABC is equal to the Square of AC , and AC is equal to BD , the Rectangle under AB, BC , shall be equal to the Square of BD . And because some Point, B , is taken without the Circle ACD , and from that Point there fall two right Lines, BCA, BD , to the Circle, one of which

cuts the Circle, and the other falls on it, and since the Rectangle under AB, BC, is equal to the Square of BD, the right Line BD shall * touch the Circle * 37. 3. ACD. And since BD touches it, and DC is drawn from the Point of Contact D, the Angle BDC is equal to the Angle in the alternate Segment of the Circle, viz. equal † to the Angle DAC. And since † 32. 3. the Angle BDC is equal to the Angle DAC; if CDA, which is common, be added, the whole Angle BDA is equal to the two Angles CDA, DAC. But the outward Angle BCD is † equal to CDA, † 32. 1. DAC. Therefore BDA is equal to BCD. But the Angle BDA * is equal to the Angle CBD, be- * 5. 1. cause the Side AD is equal to the Side AB. Wherefore DBA shall be equal to BCD: And so the three Angles BDA, DBA, BCD, are equal to each other. And since the Angle DBC is equal to the Angle BCD, the Side BD is † equal to the Side DC. But † 6. 1. BD is put equal to CA. Therefore CA is equal to CD. And so the Angle CDA is equal to the Angle DAC. Therefore the Angles CDA, DAC, taken together, are double to the Angle DAC. But the Angle BCD is equal to the Angles CDA, DAC. Therefore the Angle BCD is double to the Angle DAC. But BCD is equal to BDA, or DBA. Wherefore BDA, or DBA, is double to BDA. Therefore the Isosceles Triangle ABD is made, having one of the Angles at the Base double to the other Angle; which was to be done.

PROPOSITION XI.

PROBLEM.

To describe an equilateral and equiangular Pentagon in a given Circle.

LET ABCDE be a Circle given. It is requir'd to describe an equilateral and equiangular Pentagon in the same.

Make an Isosceles Triangle FGH, having * each * 10 of this. of the Angles at the Base GH, double to the other Angle F; and describe the Triangle ADC in the Circle ABCDE, equiangular † to the Triangle FGH; so † 2 of this. that

* 9. 1. that the Angle CAD be equal to that at F, and ACD, CDA, each equal to the Angles G or H. Wherefore the Angles ACD, CDA, are each double to the Angle CAD. This being done, bisect * ACD, CDA, by the right Lines CE, DB, and join AB, BC, DE, EA.

Then, because each of the Angles ACD, CDA, is double to CAD, and they are bisected by the right Lines CE, DB; the five Angles DAC, ACE, ECD, CDB, BDA, are equal to each other. But equal * 26. 3. Angles stand * upon equal Circumferences. Therefore the five Circumferences AB, BC, CD, DE, EA, are equal to each other. But equal Circumferences subtend † 29. equal right Lines. Therefore the five right Lines AB, BC, CD, DE, EA, are equal to each other. Wherefore ABCDE is an equilateral Pentagon. I say, it is also equiangular; for because the Circumference AB is equal to the Circumference DE, by adding the Circumference BCD, which is common, the whole Circumference ABCD is equal to the whole Circumference EDCB; but the Angle AED stands on the Circumference ABCD, and BAE on the Circumference EDCB: Therefore the Angle BAE is equal to the Angle AED. For the same Reason, each of the Angles ABC, BCD, CDE, is equal to BAE, or AED. Wherefore the Pentagon ABCDE is equiangular; but it has been prov'd to be also equilateral. And consequently there is an equilateral and equiangular Pentagon inscrib'd in a given Circle; *which was to be done.*

PROPOSITION XII.

PROBLEM.

To describe an equilateral and equiangular Pentagon about a Circle given.

LET ABCDE be the given Circle. It is requir'd to describe an equilateral and equiangular Pentagon about the same.

* By 11. of this. Let A, B, C, D, E, be the angular Points of a Pentagon suppos'd to be inscrib'd * in the Circle; so that the Circumferences AB, BC, CD, DE, EA, be equal;

equal; and let the right Lines GH, HK, KL, LM, MG, be drawn, touching † the Circle in the Points † 17. 3.
A, B, C, D, E: Let F be the Centre of the Circle ABCDE, and join FB, FK, FC, FL, FD.

Then, because the right Line KL touches the Circle ABCDE in the Point C, and the right Line FC is drawn from the Centre F to C, the Point of Contact; FC will be † perpendicular to KL: And so † 18. 3.
both the Angles at C are right Angles. For the same Reason, the Angles at the Points B, D, are right Angles. And because FCK is a right Angle, the Square of FK will be * equal to the Squares of FC, * 47. 1.
CK: And for the same Reason, the Square of FK is equal to the Squares of FB, BK. Therefore the Squares of FC, CK, are equal to the Squares of FB, BK. But the Square of FC is equal to the Square of FB. Wherefore the Square of CK shall be equal to the Square BK; and so BK is equal to CK. And because FB is equal to FC, and FK is common, the two Sides BF, FK, are equal to the two CF, FK, and the Base BK is equal to the Base KC; and so the Angle BFK shall be † equal to the Angle † 8. 1.
KFC, and the Angle BKF to the Angle FKC. Therefore the Angle BFC is double to the Angle KFC, and the Angle BKC double to the Angle FKC: For the same Reason, the Angle CFD is double to the Angle CFL, and the Angle CLD double to the Angle CLF. And because the Circumference BC is equal to the Circumference CD, the Angle BFC shall be † equal to the Angle CFD. † 27. 3.
But the Angle BFC is double to the Angle KFC, and the Angle DFC double to LFC. Therefore the Angle KFC is equal to the Angle CFL. And so FKC, FLC, are two Triangles, having two Angles of the one equal to two Angles of the other, each to each, and one Side of the one equal to one Side of the other, viz. the common Side FC; wherefore they shall have † the other Sides of the one equal † 26. 1.
to the other Sides of the other; and the other Angle of the one equal to the other Angle of the other. Therefore the right Line KC is equal to the right Line CL, and the Angle FKC to the Angle FLC. And since KC is equal to CL, KL shall be double to KC. And by the same Reason we prove, that
HK

HK is double to BK. Again, because BK has been prov'd equal to KC, and KL the double to KC, as also HK the double of BK, HK shall be equal to KL. So likewise we prove, that GH, GM, and ML, are each equal to HK, or KL. Therefore the Pentagon GHKLM is equilateral. I say also, it is equiangular; for because the Angle FKC is equal to the Angle FLC; and the Angle HKL has been prov'd to be double to the Angle FKC; and also KLM double to FLC: Therefore the Angle HKL shall be equal to the Angle KLM. By the same Reason we demonstrate, that every one of the Angles KHG, HGM, GML, is equal to the Angle HKL, or KLM. Therefore the five Angles, GHK, HKL, KLM, LMG, MGH, are equal between themselves. And so the Pentagon GHKLM is equiangular, and it has been prov'd likewise to be equilateral, and describ'd about the Circle ABCDE; *which was to be done.*

PROPOSITION XIII.

PROBLEM.

To describe a Circle in an equilateral and equiangular Pentagon.

LET ABCDE be an equilateral and equiangular Pentagon. It is requir'd to inscribe a Circle in the same.

* 9. 1.

Bisect * the Angles BCD, CDE, by the Right Lines CF, DF; and from the Point F, wherein CF, DF, meet each other, let the Right Lines FB, FA, FE, be drawn. Now, because BC is equal to CD, and CF is common, the two Sides BC, CF, are equal to the two Sides DC, CF; and the Angle BCF is equal to the Angle DCF. Therefore the Base BF is † equal to the Base FD; and the Triangle BFC equal to the Triangle DCF, and the other Angles of the one equal to the other Angles of the other, which are subtended by the equal Sides: Therefore the Angle CBF shall be equal to the Angle CDF; And because the Angle CDE is double to the Angle CDF, and the Angle CDE is equal to the Angle

† 4. 1.

Angle ABC, as also CDF, equal to CBF; the Angle CBA will be double to the Angle CBF; and so the Angle ABF equal to the Angle CBF. Wherefore the Angle ABC is bisected by the Right Line BF. After the same manner we prove, that either of the Angles BAE or AED is bisected by the Right Lines AF, FE. From the Point F draw * FG, FH, FK, * 12. 1., FL, FM, perpendicular to the Right Lines AB, BC, CD, DE, EA. Then, since the Angle HCF is equal to the Angle KCF, and the Right Angle FHC equal to the Right Angle FKC; the two Triangles FHC, FKC, shall have two Angles of the one equal to two Angles of the other, and one Side of the one equal to one Side of the other, viz. the Side FC common to each of them. And so the other Sides of the one will be † equal to the other Sides of the other, † 26. 1., and the Perpendicular FH equal to the Perpendicular FK. In the same manner we demonstrate, that FL, FM, or FG, is equal to FH, or FK. Therefore the five Right Lines FG, FH, FK, FL, FM, are equal to each other. And so a Circle describ'd on the Centre F, with either of the Distances FG, FH, FK, FL, FM, will pass thro' the other Points, and shall touch the Right Lines AB, BC, CD, DE, EA; since the Angles at G, H, K, L, M, are Right Angles: For if it does not touch them, but cuts them, a Right Line drawn from the Extremity of the Diameter of a Circle at Right Angles to the Diameter, will fall within the Circle, which is † absurd. Therefore † 16. 3., a Circle describ'd on the Centre F, with the Distance of any one of the Points G, H, K, L, M, will not cut the Right Lines AB, BC, CD, DE, EA; and so will necessarily touch them; *which was to be done.*

Coroll. If two of the nearest Angles of an equilateral and equiangular Figure be bisected, and, from the Point in which the Lines bisecting the Angles meet, there be drawn Right Lines to the other Angles of the Figure, all the Angles of the Figure will be bisected.

PROPOSITION XIV.

PROBLEM.

To describe a Circle about a given equilateral and equiangular Pentagon.

LET $ABCDE$ be an equilateral and equiangular Pentagon. It is requir'd to describe a Circle about the same.

* Cor. of
preced.

† 6. 1.

Bisect both the Angles BCD , CDE , by the right Lines CF , FD ; and draw FB , FA , FE , from the Point F , in which they meet. Then each of the Angles CBA , BAE , AED , shall be bisected * by the right Lines BF , FA , FE . And since the Angle BCD is equal to the Angle CDE ; and the Angle BCD is half the Angle BCD , as likewise CDF half CDE ; the Angle BCD will be equal to the Angle FCD ; and so the Side CF † equal to the Side FD . We demonstrate, in like manner, that FB , FA , or FE , is equal to FC , or FD . Therefore the five right Lines FA , FB , FC , FD , FE , are equal to each other. And so a Circle being describ'd on the Centre F , with any of the Distances FA , FB , FC , FD , FE , will pass through the other Points, and will be describ'd about the equilateral and equiangular Pentagon $ABCDE$; *which was to be done.*

PROPOSITION XV.

PROBLEM.

To inscribe an equilateral and equiangular Hexagon in a given Circle.

LET $ABCDEF$ be a Circle given. It is requir'd to inscribe an equilateral and equiangular Hexagon therein.

Draw AD , a Diameter of the Circle $ABCDEF$, and let G be the Centre; and about the Point D , as a Centre, with the Distance DG , let a Circle $EGCH$, be describ'd; join EG , GC , which produce to the Points B , F : Likewise join AB , BC , CD , DE , EF , FA .

FA. I say, ABCDEF is an equilateral and equiangular Hexagon.

For, since the Point G is the Centre of the Circle ABCDEF, GE will be equal to GD. Again, because the Point D is the Centre of the Circle EGCH, DE shall be equal to DG: But GE has been prov'd equal to GD. Therefore GE is equal to ED. And so EGD is an equilateral Triangle; and consequently the three Angles thereof, EGD, GDE, DEG, are * equal between themselves: But the * Cor. 5. 1. three Angles of a Triangle are † equal to two right † 32. 1. Angles. Therefore the Angle EGD is a third Part of two right Angles. In the same manner we demonstrate, that DGC is one third Part of two right Angles: And since the right Line CG, standing upon the right Line EB, makes † the adjacent Angles † 13. 1. EGC, CGB; the other Angle, CGB, is also one third Part of two right Angles. Therefore the Angles EGD, DGC, CGB, are equal between themselves: And the Angles that are vertical to them, viz. the Angles BGA, AGF, FGE, are * equal to the * 15. 1. Angles EGD, DGC, CGB. Wherefore the six Angles EGD, DGC, CGB, BGA, AGF, FGE, are equal to one another. But equal Angles stand † on † 26. 3. equal Circumferences. Therefore the six Circumferences, AB, BC, CD, DE, EF, FA, are equal to each other. But equal right Lines subtend † equal † 29. 3. Circumferences. Therefore the six right Lines are equal between themselves; and accordingly the Hexagon ABCDEF is equilateral: I say, it is also equiangular. For, because the Circumference AF is equal to the Circumference ED, add the common Circumference ABCD, and the whole Circumference FABCD is equal to the whole Circumference EDCBA. But the Angle FED stands on the Circumference FABCD; and the Angle AFE, on the Circumference EDCBA. Therefore the Angle AFE is * equal to the Angle DEF. In the same * 27. 3. manner we prove, that the other Angles of the Hexagon ABCDEF are severally equal to AFE, or FED. Therefore the Hexagon ABCDEF is equiangular. But it has been prov'd to be also equilateral, and is inscrib'd in the Circle ABCDEF; which was to be done.

Coroll. From hence it is manifest, that the Side of the Hexagon is equal to the Semidiameter of the Circle. And if we draw thro' the Points A, B, C, D, E, F, Tangents to the Circle, an equilateral and equiangular Hexagon will be describ'd about the Circle, as is manifest, from what has been said concerning the Pentagon. And so likewise may a Circle be inscrib'd and circumscrib'd about a given Hexagon; *which was to be done.*

PROPOSITION XVI.

PROBLEM.

To describe an equilateral and equiangular Quindecagon in a given Circle.

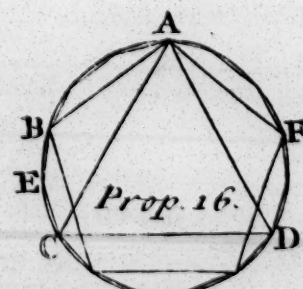
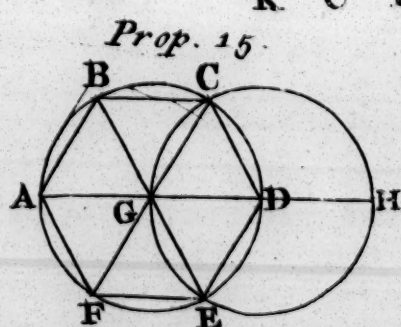
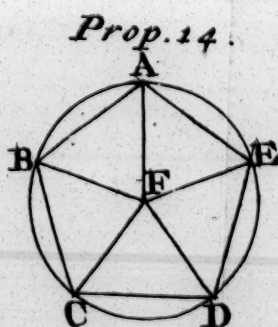
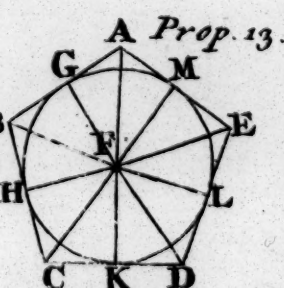
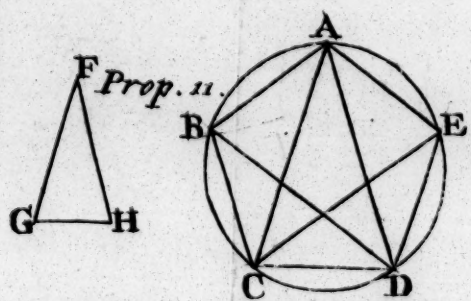
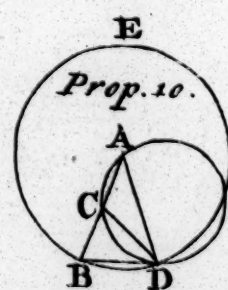
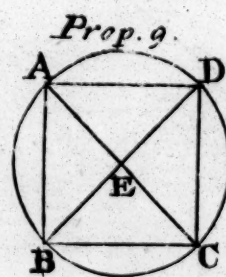
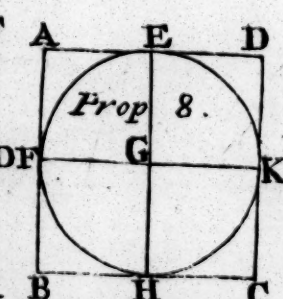
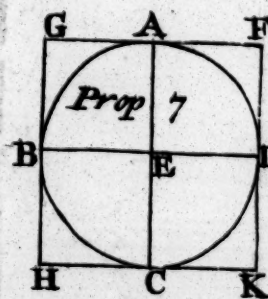
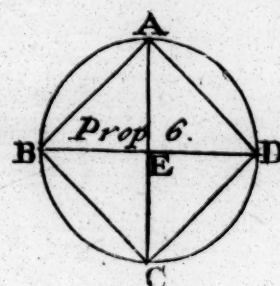
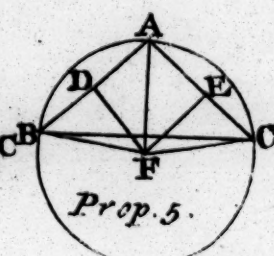
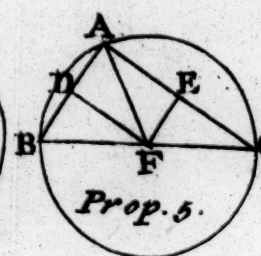
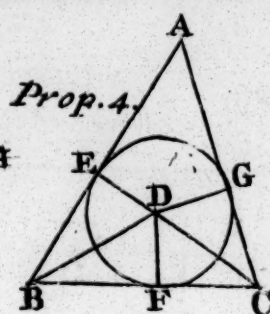
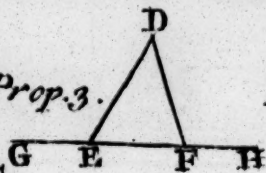
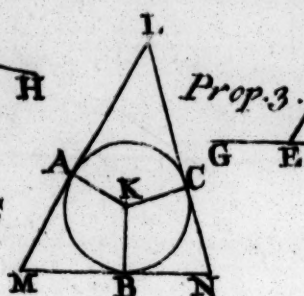
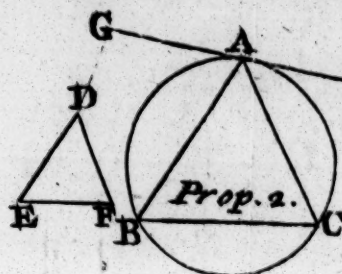
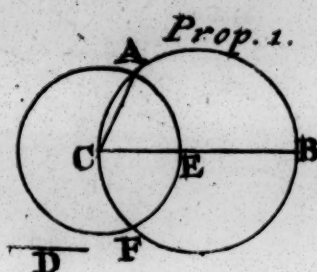
LET ABCD be a Circle given. It is requir'd to describe an equilateral and equiangular Quindecagon in the same.

Let AC be the Side of an equilateral Triangle inscrib'd in the Circle ABCD, and AB the Side of a Pentagon. Now, if the whole Circumference of the Circle ABCD be divided into fifteen equal Parts, the Circumference ABC, one Third of the Whole, shall be five of the said fifteen equal Parts; and the Circumference AB, one Fifth of the Whole, will be three of the said Parts. Wherefore the remaining Circumference BC will be two of the said Parts. And if BC be bisected in the Point E, BE, or EC, will be one fifteenth Part of the whole Circumference ABCD. And so, if BE, EC, be join'd, and either EC, or EB, be continually applied in the Circle, there shall be an equilateral and equiangular Quindecagon describ'd in the Circle ABCD; *which was to be done.*

If, according to what has been said of the Pentagon, right Lines are drawn thro' the Divisions of the Circle touching the same, there will be describ'd about the Circle an equilateral and equiangular Quindecagon. And, moreover, a Circle may be inscrib'd, or circumscrib'd, about a given equilateral and equiangular Quindecagon.

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Book 4.



2



E U C L I D's ELEMENTS.

B O O K V.

DEFINITIONS.

- I. **A** Part is a Magnitude of a Magnitude, the Less of the Greater, when the Lesser measures the Greater.
- II. But a Multiple is a Magnitude of a Magnitude, the Greater of the Lesser, when the Lesser measures the Greater.
- III. Ratio is a certain mutual Habitude of Magnitudes of the same Kind, according to Quantity.
- IV. Magnitudes are said to have Proportion to each other, which, being multiplied, can exceed one another.
- V. Magnitudes are said to be in the same Ratio, the first to the second, and the third to the fourth, when the Equimultiples of the first and third, compared with the Equimultiples of the second and fourth, according to any Multiplication whatsoever, are either both together greater, equal, or less than the Equimultiples of the second and fourth, if those be taken that answer each other.

That is, if there be four Magnitudes, and you take any Equimultiples of the first and third, and also any Equimultiples of the second and fourth; and if the Multiple of the first be greater than the Multiple of the second, and also the Multiple of the third greater than the Multiple of the fourth: Or, if the Multiple of the first be equal to the Multiple of the second; and also the Multiple of the third equal to the Multiple of the fourth: Or, lastly, if the Multiple of the first be less than the Multiple of the second; and also that of the third less than that of the fourth, and these Things happen according to every Multiplication whatsoever; then the four Magnitudes are in the same Ratio, the first to the second, as the third to the fourth.

VI. *Magnitudes that have the same Proportion, are called Proportionals.*

Expounders usually lay down here that Definition which *Euclid* has given for Numbers only, in his Seventh Book; viz. That

Magnitudes are said to be Proportionals, when the first is the same Equimultiple of the second, as the third is of the fourth, or the same Part or Parts.

But this Definition appertains only to Numbers, and commensurable Quantities; and so, since it is not universal, *Euclid* did well to reject it in this Element, which treats of the Properties of all Proportionals; and to substitute another general one, agreeing to all Kinds of Magnitudes. In the mean time, Expounders very much endeavour to demonstrate the Definition here laid down by *Euclid*, by the usual receiv'd Definition of proportional Numbers; but this much easier flows from that, than that from this; which may be thus demonstrated:

First, Let A, B, C, D, be four Magnitudes, which are in the same Ratio, according to the Conditions that Magnitudes in the same Ratio must have laid down in the fifth Definition. And let the first be a Multiple of the second. I say, the third is also the same Multiple of the fourth. For Example: Let A be

be equal to $5B$. Then C shall be equal to $5D$. Take any Number, for Example, 2, by which let 5 be multiplied, and the Product will be 10: And let $2A$, $2C$, be Equimultiples of the first and third Magnitudes A and C : Also, let $10B$ and $10D$ be Equimultiples of the second and fourth Magnitudes B and D . Then (by *Def. 5.*) if $2A$ be equal to $10B$, $2C$ shall be equal to $10D$. But since A (from the *Hypothesis*) is five times B , $2A$ shall be equal to $10B$; and so $2C$ equal to $10D$, and C equal to $5D$; that is, C will be five times D . W. W. D.

Secondly, Let A be any Part of B ; then C will be the same Part of D . For, because A is to B , as C is to D ; and since A is some Part of B ; then B will be a Multiple of A : And so (by *Case 1.*) D will be the same Multiple of C , and accordingly C shall be the same Part of the Magnitude D , as A is of B . W. W. D.

Thirdly, Let A be equal to any Number of whatsoever Parts of B . I say, C is equal to the same Number of the like Parts of D . For Example: Let A be a fourth Part of five times B ; that is, let A be equal to $\frac{1}{4}5B$. I say, C is also equal to $\frac{1}{4}5D$. For, because A is equal to $\frac{1}{4}5B$, each of them being multiplied by 4, then $4A$ will be equal to $5B$. And so, if the Equimultiples of the first and third, *viz.* $4A$, $4C$, be assumed; as also the Equimultiples of the second and fourth, *viz.* $5B$, $5D$: And (by the *Definition*) if $4A$ is equal to $5B$; then $4C$ is equal to $5D$. But $4A$ has been proved equal to $5B$, and so $4C$ shall be equal to $5D$, and C equal to $\frac{1}{4}5D$. W. W. D.

And universally, if A be equal to $\frac{n}{m}B$, C will be equal to $\frac{n}{m}D$. For let A and C be multiplied by m , and B and D by n . And because A is equal to $\frac{n}{m}B$; mA shall be equal to nB ; wherefore (by *Def. 5.*) mC will be equal to nD , and C equal to $\frac{n}{m}D$. W. W. D.

VII. When of Equimultiples, the Multiple of the first exceeds the Multiple of the second, but the Multiple of the third does not exceed the Multiple of the fourth; then the first to the second is said to have a greater Proportion than the third to the fourth.

VIII. Analogy is a Similitude of Proportions.

IX. Analogy at least consists of three Terms.

X. When three Magnitudes are Proportionals, the first is said to have, to the third, a duplicate Ratio to what it has to the second.

XI. But when four Magnitudes are Proportionals, the first shall have a triplicate Ratio to the fourth of what it has to the second; and so always one more in Order, as the Proportionals shall be extended.

XII. Homologous Magnitudes, or Magnitudes of a like Ratio, are said to be such whose Antecedents are to the Antecedents, and Consequents to the Consequents.

XIII. Alternate Ratio is the comparing of the Antecedent with the Antecedent, and the Consequent with the Consequent.

XIV. Inverse Ratio is when the Consequent is taken as the Antecedent, and so compared with the Antecedent as a Consequent.

XV. Compounded Ratio is when the Antecedent and Consequent, taken both as one, is compared to the Consequent itself.

XVI. Divided Ratio is when the Excess, wherein the Antecedent exceeds the Consequent, is compared with the Consequent.

XVII. Converse Ratio is when the Antecedent is compared with the Excess, by which the Antecedent exceeds the Consequent.

XVIII. Ratio of Equality is where there are taken more than two Magnitudes in one Order, and a like

like Number of Magnitudes in another Order, comparing two to two being in the same Proportion; and it shall be in the first Order of Magnitude, as the first is to the last, so in the second Order of Magnitudes is the first to the last: Or otherwise, it is the Comparison of the Extremes together, the Means being omitted.

XIX. Ordinate Proportion is, when as the Antecedent is to the Consequent, so is the Antecedent to the Consequent; and as the Consequent is to any other, so is the Consequent to any other.

XX. Perturbate Proportion is when there are three Magnitudes, and others also, that are equal to these in Multitude, as in the first Magnitudes the Antecedent is to the Consequent; so in the second Magnitude is the Antecedent to the Consequent: And as in the first Magnitudes the Consequent is to some other, so in the second Magnitudes, is some other to the Antecedent.

A X I O M S.

- I.** **E**quimultiples of the same, or of equal Magnitudes, are equal to each other.
- II.** Those Magnitudes that have the same Equimultiple, or whose Equimultiples are equal, are equal to each other.

PROPOSITION I.

THEOREM.

If there be any Number of Magnitudes Equimultiples of a like Number of Magnitudes, each to each; whatsoever Multiple any one of the former Magnitudes is of its correspondent one, the same Multiple are all the former Magnitudes of all the latter.

LET there be any Number of Magnitudes, AB, CD, Equimultiples of a like Number of Magnitudes E, F, each of each. I say, what Multiple the Magnitude AB is of E, the same Multiple AB, and CD, together, is of E and F together.

For, because AB and CD are Equimultiples of E and F, as many Magnitudes equal to E, that are in AB, so many shall be equal to F in CD. Now, divide AB into Parts equal to E, which let be AG, GB; and CD into Parts equal to F, viz. CH, HD. Then the Multitude of Parts, CH, HD, shall be equal to the Multitude of Parts AG, GB. And since AG is equal to E, and CH to F; AG and CH, together, shall be equal to E and F together. By the same Reason, because GB is equal to E, and HD to F, GB and HD will be equal to E and F together. Therefore, as often as E is contain'd in AB, so often is E and F contain'd in AB and CD.

And so, as often as E is contain'd in AB, so often are E and F, together, contain'd in AB and CD together. Therefore, if there are any Number of Magnitudes Equimultiples of a like Number of Magnitudes, each to each; whatsoever Multiple any one of the former Magnitudes is of its correspondent one, the same Multiple are all the former Magnitudes of all the latter; which was to be demonstrated.

P R O-

PROPOSITION II.

THEOREM.

If the first be the same Multiple of the second, as the third is of the fourth; and if the fifth be the same Multiple of the second, as the sixth is of the fourth; then shall the first, added to the fifth, be the same Multiple of the second, as the third, added to the sixth, is of the fourth.

LET the first AB be the same Multiple of the second C , as the third DE is of the fourth F ; and let the fifth BG be the same Multiple of the second C , as the sixth EH is of the fourth F . I say, the first added to the fifth, *viz.* AG , is the same Multiple of the second C , as the third added to the sixth, *viz.* DH , is of the fourth F .



For, because AB is the same Multiple of C , as DE is of F ; there are as many Magnitudes equal to C in AB , as there are Magnitudes equal to F in DE . And for the same Reason, there are as many Magnitudes equal to C in BG , as there are Magnitudes equal to F in EH . Therefore there are as many Magnitudes equal to C , in the whole AG , as there are Magnitudes equal to F in DH . Wherefore AG is the same Multiple of C , as DH is of F . And so the first, added to the fifth AG , is the same Multiple of the second C , as the third, added to the sixth DH , is of the fourth F . Therefore, *if the first be the same Multiple of the second, as the third is of the fourth; and if the fifth be the same Multiple of the second, as the sixth is of the fourth; then shall the first, added to the fifth, be the same Multiple of the second, as the third, added to the sixth, is of the fourth; which was to be demonstrated.*

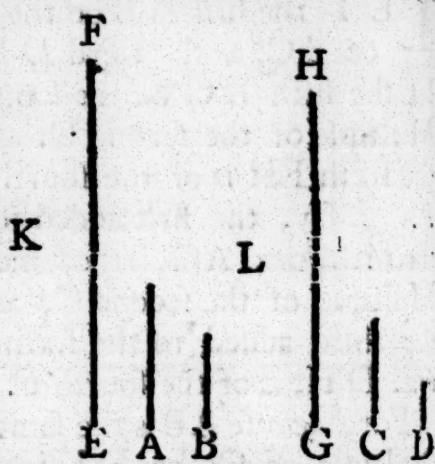
PROPOSITION III.

THEOREM.

If the first be the same Multiple of the second, as the third is of the fourth, and there be taken Equimultiples of the first and third; then will each of the Magnitudes taken be Equimultiples of the second and fourth.

LET the first A be the same Multiple of the second B, as the third C is of the fourth D; and let EF, GH, be Equimultiples of A and C. I say, EF is the same Multiple of B, as GH is of D.

For, because EF is the same Multiple of A, as GH is of C, there are as many Magnitudes equal to A in EF, as there are Magnitudes equal to C in GH.



Now, divide EF into the Magnitudes EK, KF, equal to A, and GH into the Magnitudes GL, LH, equal to C. Then the Number of the Magnitudes EK, KF, will be equal to the Number of the Magnitudes GL, LH. And because A is the same Multiple of B, as C is of D, and EK is equal to A, and GL to C; EK will be the same Multiple of B, as GL is of D. For the same Reason, KF shall be the same Multiple of B, as LH is of D. Therefore, because the first EK is the same Multiple of the second B, as the third GL is of the fourth D, and KF, LH, are Equimultiples of the second B and fourth D; the first added to the fifth EF, shall be * the same Multiple of the second B, as the third added to the sixth GH is of the fourth D. If, therefore, the first be the same Multiple of the second, as the third is of the fourth, and there be taken Equimultiples of the first and third; then will each of the Magnitudes taken be Equimultiples of the second and fourth; which was to be demonstrated.

* 2 of this.

P R O-

PROPOSITION IV.

THEOREM.

If the first have the same Proportion to the second, as the third to the fourth, then also shall the Equimultiples of the first and third have the same Proportion to the Equimultiples of the second and fourth, according to any Multiplication whatsoever, if they be so taken as to answer each other.

LET the first A have the same Proportion to the second B, as the third C hath to the fourth D; and let E and F, the Equimultiples of A and C, be any-how taken; as also G, H, the Equimultiples of B and D. I say, E is to G as F is to H.

For take K and L, any Equimultiples of E and F; and also M and N of G and H.

Then, because E is the same Multiple of A, as F is of C, and K, L, are taken Equimultiples of E, F, K, will be * the same Multiple of A, as L is of C. For the same Reason, M is the same Multiple of B, as N is of D. And since A is to B, as C is to D, and K and L are Equimultiples of A and C; and also M and N Equimultiples of B and D; if K exceeds M, then † L will exceed N; if equal, equal; or less, less. And K, L, are Equimultiples of E, F, and M, N, any other Equimultiples of GH.

Therefore, as E is to G, so



† Def. 5.

shall † F be to H. Wherefore, if the first have the same Proportion to the second, as the third to the fourth; then also shall the Equimultiples of the first and third have the same Proportion to the Equimultiples of the second and fourth, according to any Multiplication whatsoever, if they be so taken as to answer each; which was to be demonstrated.

Because it is demonstrated, if K exceeds M, then L will exceed N; and if it be equal to it, it will be equal; and if less, lesser. It is manifest, likewise, if M exceeds K, that N shall exceed L; if equal, equal; but if less, less. And therefore as G is to E, so is

* Def. 5.

* H to F.

Coroll. From hence it is manifest, if four Magnitudes be proportional, that they will be also inversely proportional.

PROPOSITION V.

THEOREM.

If one Magnitude be the same Multiple of another Magnitude, as a Part taken from the one is of a Part taken from the other; then the Residue of the one shall be the same Multiple of the Residue of the other, as the Whole is of the Whole.

LET the Magnitude AB be the same Multiple of the Magnitude CD, as the Part taken away AE is of the Part taken away CF. I say, that the Residue EB is the same Multiple of the Residue FD, as the whole AB is of the whole CD.

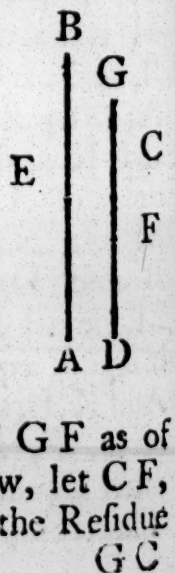
For, let EB be such a Multiple of CG, as AE is of CF.

Then, because AE is the same Multiple of CF, as EB is of CG, AE will be * the same Multiple of CF, as AB is of GF. But AE and AB are put Equimultiples of CF and CD.

* 1 of this.

† 2 Axiom of this.

Therefore AB is the same Multiple of GF as of CD; and so GF is † equal to CD. Now, let CF, which is common, be taken away; and the Residue



GC is equal to the Residue DF. And then, because AE is the same Multiple of CF, as EB is of CG, and CG is equal to DF; AE shall be the same Multiple of CF, as EB is of FD. But AE is put the same Multiple of CF, as AB is of CD. Therefore EB is the same Multiple of FD, as AB is of CD; and so the Residue EB is the same Multiple of the Residue FD, as the whole AB is of the whole CD. Wherefore, *if one Magnitude be the same Multiple of another Magnitude, as a Part taken from the one is of a Part taken from the other; then the Residue of the one shall be the same Multiple of the Residue of the other, as the Whole is of the Whole*; which was to be demonstrated.

PROPOSITION VI.

THEOREM.

If two Magnitudes be Equimultiples of two Magnitudes, and some Magnitudes, Equimultiples of the same, be taken away; then the Residues are either equal to those Magnitudes, or else Equimultiples of them.

LET two Magnitudes AB, CD, be Equimultiples of two Magnitudes E, F; and let the Magnitudes AG, CH, Equimultiples of the same E, F, be taken from AB, CD. I say, the Residues GB, HD, are either equal to E, F, or are Equimultiples of them.

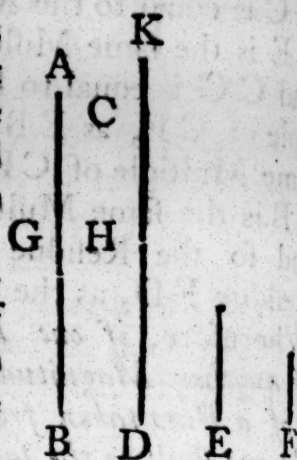
For, first, Let GB be equal to E. I say, HD is also equal to F. For let CK be equal to F. Then, because AG is the same Multiple of E, as CH is of F; and GB is equal to E; and CK to F; AB will be * the same Multiple of E, as KH is of F. But AB and CD are put Equimultiples of E and F. Therefore KH is the same Multiple of F, as CD is of F.

And because KH and CD are Equimultiples of F; KH will be equal to CD. Take



* 1 of this.

Take away CH , which is common; then the Residue KC is equal to the Residue HD . But KC is equal to F . Therefore HD is equal to F ; and so GB shall be equal to E , and HD to F .



In like manner we demonstrate, if GB was a Multiple of E , that HD is the like Multiple of F . Therefore, if two Magnitudes be Equimultiples of two Magnitudes, and some Magnitudes, Equimultiples of the same, be taken away; then the Residues are either equal to those Magnitudes, or else Equimultiples of them; which was to be demonstrated.

PROPOSITION VII.

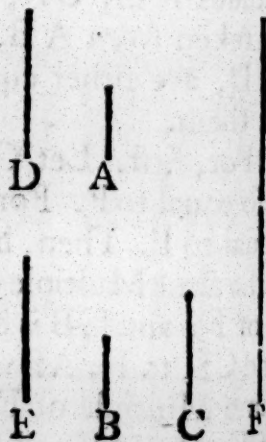
PROBLEM.

Equal Magnitudes have the same Proportion to the same Magnitude; and one and the same Magnitude has the same Proportion to equal Magnitudes.

LET A, B , be equal Magnitudes, and let C be any other Magnitude. I say, A and B have the same Proportion to C ; and likewise C has the same Proportion to A as to B .

For take D, E , Equimultiples of A and B ; and let F be any other Multiple of C .

Now, because D is the same Multiple of A , as E is of B , and A is equal to B , D shall be also equal to E ; but F is a Magnitude taken at Pleasure. Therefore, if D exceeds F , then E will exceed F ; if D be equal to F , E will be equal to F ; and if less, less. But D, E , are Equimultiples of A, B ; and F is any Multiple of C . Therefore it will be * as A is to C , so is B to C .



* Def. 5.

I say,

I say, moreover, that C has the same Proportion to A as to B. For the same Construction remaining, we prove, in like manner, that D is equal to E. Therefore, if F exceeds D, it will also exceed E; if it be equal to D, it will be equal to E; and if it be less than D, it will be less than E. But if F is Multiple of C; and D, E, any other Equimultiples of A, B; therefore as C is to A, so shall * C be to B. Wherefore * *Def. 5.* equal Magnitudes have the same Proportion to the same Magnitude, and the same Magnitude to equal ones; which was to be demonstrated.

PROPOSITION VIII.

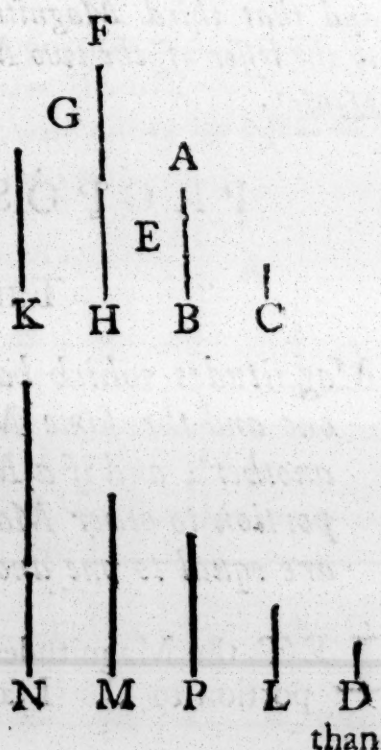
THEOREM.

The greater of any two unequal Magnitudes has a greater Proportion to some third Magnitude, than the less has; and that third Magnitude hath a greater Proportion to the lesser of the two Magnitudes, than it has to the greater.

LET AB and C be two unequal Magnitudes, whereof AB is the greater; and let D be any third Magnitude. I say, AB has a greater Proportion to D, than C has to D; and D has a greater Proportion to C, than it has to AB.

Because AB is greater than C, make BE equal to C, that is, let AB exceed C by AE; then AE, multiplied some Number of Times, will be greater than D. Now let AE be multiplied until it exceeds D, and let that Multiple of AE, greater than D, be FG. Make GH the same Multiple of EB, and K of C, as FG is of AE. Also, assume L double to D, P triple, and so on, until such a Multiple of D is had, as is the nearest greater than K; let this be N, and let M be a Multiple of D the nearest less than N.

Now, because N is the nearest Multiple of D greater



than

than K, M will not be greater than K; that is, K will not be less than M. And since FG is the same Multiple of AE, as GH is of EB; FG shall be * the same Multiple of AE, as FH is of AB; but FG is the same Multiple of AE, as K is of C; wherefore FH is the same Multiple of AB, as K is of C; that is, FH, K, are Equimultiples of AB and C. Again, because GH is the same Multiple of EB, as K is of C, and EB is equal to C; GH shall be † equal to K. But K is not less than M. Therefore GH shall not be less than M; but FG is greater than D. Therefore the whole FH will be greater than M and D; but M and D together are equal to N, because M is a Multiple of D, the nearest lesser than N: Wherefore FH is greater than N. And so, since FH exceeds N, and K does not, and FH and K are Equimultiples of AB and C, and N is another Multiple of D; therefore AB will have ‡ a greater Ratio to D, than C has to D. I say, moreover, that D has a greater Ratio to C, than it has to AB; for the same Construction remaining, we demonstrate, as before, that N exceeds K, but not FH. And N is a Multiple of D, and FH, K, are Equimultiples of AB and C. Therefore D has ‡ a greater Proportion to C, than D hath to B. Wherefore *the greater of any two unequal Magnitudes has a greater Proportion to some third Magnitude, than the less has; and that third Magnitude hath a greater Proportion to the lesser of the two Magnitudes, than it has to the greater.*

PROPOSITION IX.

THEOREM.

Magnitudes which have the same Proportion to one and the same Magnitude, are equal to one another; and if a Magnitude has the same Proportion to other Magnitudes, these Magnitudes are equal to one another.

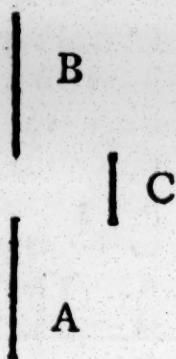
LET the Magnitudes A and B have the same Proportion to C. I say, A is equal to B.

For,

For, if it was not, A and B would not * have the * 8 of this. same Proportion to the same Magnitude C; but they have. Therefore A is equal to B.

Again, let C have the same Proportion to A as to B. I say, A is equal to B.

For, if it be not, C will not have the same Proportion to A as to B; but it hath: Therefore A is necessarily equal to B. Therefore, *Magnitudes that have the same Proportion to one and the same Magnitude, are equal to one another; and, if a Magnitude has the same Proportion to other Magnitudes, these Magnitudes are equal to one another; which was to be demonstrated.*



PROPOSITION X.

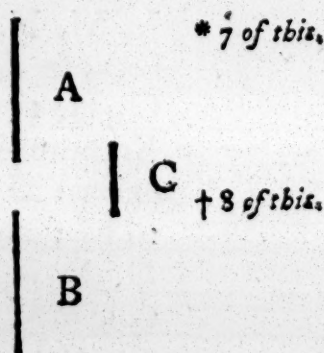
THEOREM.

Of Magnitudes having Proportion to the same Magnitude, that which has the greater Proportion, is the greater Magnitude: And the Magnitude to which the same bears a greater Proportion, is the lesser Magnitude.

LET A have a greater Proportion to C, than B has to C. I say, A is greater than B.

For, if it be not greater, it will either be equal or less. But A is not equal to B, because then both A and B would have * the same Proportion to the same Magnitude C; but they have not. Therefore A is not equal to B: Neither is it less than B; for then A would have † a less Proportion to C, than B would have; but it hath not a less Proportion: Therefore A is not less than B. But it has been prov'd likewise not to be equal to it: Therefore A shall be greater than B.

Again, let C have a greater Proportion to B than to A. I say, B is less than A.



K

For,

For if it be not less, it is greater or equal. Now, ** 7 of this.* B is not equal to A, for then C would have *** the same Proportion to A as to B; but this it has not. Therefore A is not equal to B; neither is B greater than A; for if it was, C would have a less Proportion to B than to A; but it has not: Therefore B is not greater than A. But it has also been prov'd not to be equal to it. Wherefore B shall be less than A. Therefore, *of Magnitudes having Proportion to the same Magnitude, that which has the greater Proportion, is the greater Magnitude: And that Magnitude to which the same bears a greater Proportion, is the lesser Magnitude;* which was to be demonstrated.

PROPOSITION XI.

THEOREM.

Proportions that are one and the same to any third, are also the same to one another.

LET A be to B, as C is to D; and C to D, as E to F. I say, A is to B, as E is to F.

For, take G, H, K, Equimultiples of A, C, E; and

G—————	H—————	K—————
A—————	C—————	E—————
B—————	D—————	F—————
L—————	M—————	N—————

** 3 Def. of this.* L, M, N, other Equimultiples of B, D, F. Then, because A is to B, as C is to D, and there are taken G, H, the Equimultiples of A and C, and L, M, any Equimultiples of B, D; if G exceeds L, *** then H will exceed M; and if G be equal to L, H will be equal to M; and if less, lesser. Again, because as C is to D, so is E to F; and H and K are taken Equimultiples of C and E; as likewise M, N, any Equimultiples of D, F; if H exceeds M, *** then K will exceed N; and if H be equal to M, K will be equal to N; and if less, lesser. But if H exceeds M, G will also exceed L; if equal, equal; and if less, less. Wherefore, if G exceeds L, K will also exceed N; and if

G be equal to L, K will be equal to N; and if less, less. But G, K, are Equimultiples of A, E; and L, N, are Equimultiples of B, F. Consequently, as A is to B, so * is E to F. Therefore, *Proportions that are * Def. 5. of one and the same to any Third, are also the same to one^{this} another*; which was to be demonstrated.

PROPOSITION XII.

THEOREM.

If any Number of Magnitudes be proportional, as one of the Antecedents is to one of the Consequents, so are all the Antecedents to all the Consequents.

LET there be any Number of proportional Magnitudes, A, B, C, D, E, F; whereof as A is to B,

G_____	H_____	K_____
A_____	C_____	E_____
B_____	D_____	F_____
L_____	M_____	N_____

so C is to D, and so E to F. I say, as A is to B, so are all the Antecedents A, C, E, to all the Consequents B, D, F.

For, let G, H, K, be Equimultiples of A, C, E; and L, M, N, any Equimultiples of B, D, F.

Then, because as A is to B, so is C to D, and so E to F; and G, H, K, are Equimultiples of A, C, E; and L, M, N, Equimultiples of B, D, F; if G exceeds L, H * will also exceed M, and K * *Def. 5. of* will exceed N; if G be equal to L, H will be equal^{this} to M, and K to N; and if less, less. Wherefore, also, if G exceeds L, then G, H, K, together, will likewise exceed L, M, N, together; and, if G be equal to L, then G, H, K, together, will be equal to L, M, N, together; and if less, less: But G, and G, H, K, are Equimultiples of A; and A, C, E; because, if there are any Number of Magnitudes Equimultiples to a like Number of Magnitudes, each to the other, the same Multiple that one Magnitude is of one, so shall † all the Magnitudes be of all. † *† of this.* And, for the same Reason, L, and L, M, N, are

K 2

Equi-

* 5 Def. of this.

Equimultiples of B, and B, D, F. Therefore, as A is to B, so * is A, C, E, to B, D, F. Wherefore, if there be any Number of Magnitudes proportional, as one of the Antecedents is to one of the Consequents, so are all the Antecedents to all the Consequents; which was to be demonstrated.

PROPOSITION XIII.

THEOREM.

If the first has the same Proportion to the second, as the third to the fourth; and if the third has a greater Proportion to the fourth, than the fifth to the sixth; then also shall the first have a greater Proportion to the second, than the fifth has to the sixth.

LET the first A have the same Proportion to the second B, as the third C has to the fourth D; and let the third C have a greater Proportion to the fourth D, than the fifth E to the sixth F. I say,

M_____	G_____	H_____
A_____	C_____	E_____
B_____	D_____	F_____
N_____	K_____	L_____

Likewise, that the first A to the second B has a greater Proportion, than the fifth E to the sixth F.

* 7 Def. of this.

For, because C has a greater Proportion to D, than E has to F; there are * certain Equimultiples of C and E, and others of D and F, such that the Multiple of C may exceed the Multiple of D; but the Multiple of E not that of F. Now let these Equimultiples of C and E, be G and H; and K and L, those of D and F; so that G exceeds K, and H not L: Make M the same Multiple of A, as G is of C; and N the same of B, as K is of D.

† 5 Def.

Then, because A is to B, as C is to D; and M and G are Equimultiples of A, C; and N, K, of B, D: If M exceeds N, then † G will exceed K; and if M be equal to N, G will be equal to K; and if less, less. But G does exceed K. Therefore M will also

so exceed N. But H does not exceed L. And M, H, are Equimultiples of A, E; and N, L, any others of B, F. Therefore A has a * greater Proportion to B, than E has to F. Wherefore, *if the first has the same Proportion to the second, as the third to the fourth; and if the third has a greater Proportion to the fourth, than the fifth to the sixth; then, also, shall the first have a greater Proportion to the second, than the fifth has to the sixth; which was to be demonstrated.*

PROPOSITION XIV.

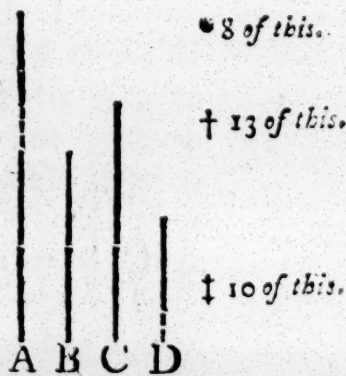
THEOREM.

If the first has the same Proportion to the second, as the third has to the fourth; and if the first be greater than the third; then will the second be greater than the fourth. But, if the first be equal to the third, then the second shall be equal to the fourth; and, if the first be less than the third, then the second will be less than the fourth.

LET the first A have the same Proportion to the second B, as the third C has to the fourth D: And let A be greater than C. I say, B is also greater than D.

For, because A is greater than C, and B is any other Magnitude: A will have * greater Proportion to B, than C has to B; but as A is to B, so is C to D; therefore, also, C shall † have a greater Proportion to D, than C hath to B. But that Magnitude to which the same bears a greater Proportion, is ‡ the lesser Magnitude: Wherefore D is less than B; and consequently B will be greater than D. In

like manner we demonstrate, if A be equal to C, that B will be equal to D; and if A be less than C, that B will be less than D. Therefore, *if the first has the same Proportion to the second, as the third has to the fourth; and if the first be greater than the third; then will the second be greater than*



the fourth. But if the first be equal to the third, then the second shall be equal to the fourth; and if the first be less than the third, then the second will be less than the fourth; which was to be demonstrated.

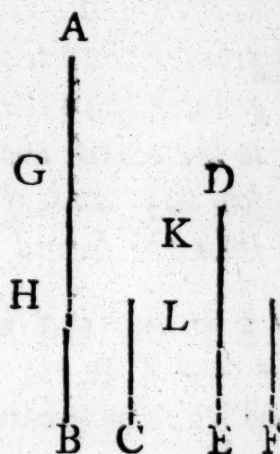
PROPOSITION XV.

THEOREM.

Parts have the same Proportion as their like Multiples, if taken correspondently.

LET AB be the same Multiple of C, as DE is of F. I say, as C is to F, so is AB to DE.

For, because AB and DE are Equimultiples of C and F, there shall be as many Magnitudes equal to C in AB, as there are Magnitudes equal to F in DE. Now, let AB be divided into the Magnitudes AG, GH, HB, each equal to C; and ED into the Magnitudes DK, KL, LE, each equal to F; then the Number of the Magnitudes AG, GH, HB, will be equal to the Number of the Magnitudes DK, KL,



LE. Now, because AG, GH, HB, are equal, as likewise DK, KL, LE, it shall be * as AG is to DK, so is GH to KL, and so is HB to LE. But as one of the Antecedents is to one of the Consequents, so † all the Antecedents to all the Consequents. Therefore, as AG is to DK, so is AB to DE. But AG is equal to C, and DK to F. Whence, as C is to F, so shall AB be to DE. Therefore, *Parts have the same Proportion as their like Multiples, if taken correspondently*; which was to be demonstrated.

* 7 of this.

† 12 of this.

PROPOSITION XVI.

THEOREM.

If four Magnitudes of the same Kind are proportional, they shall also be alternately proportional.

LET four Magnitudes A, B, C, D, be proportional; whereof A is to B, as C is to D. I say, likewise, that they will be alternately proportional, *viz.* as A is to C, so is B to D; for take E, F, Equimultiples of A and B; and G,

H, any Equimultiples of C, D.

Then, because E is the same Multiple of A, as F is of B, and

E	_____	G	_____
A	_____	C	_____
B	_____	D	_____
F	_____	H	_____

Parts have the same Proportion * to their like Multiples, if taken correspondently; it shall be, as A is to B, so is E to F. But as A is to B, so is C to D.

Therefore, also, as C is to D, so † is E to F. Again, † 11 of this, because G, H, are Equimultiples of C and D, and

Parts have the same Proportion with their like Multiples, if taken correspondently, it will be, as C is to D, so is G to H; but as C is to D, so is E to F.

Therefore, also, as E is to F, so is G to H; and if four Magnitudes be proportional, and the first greater than the third, then the second will be ‡ greater than ‡ 14 of this, the fourth; and if the first be equal to the third, the

second will be equal to the fourth; and if less, less.

Therefore, if E exceeds G, F will exceed H; and if E be equal to G, F will be equal to H; and if less, less.

But E, F, are any Equimultiples of A, B; and G, H, any Equimultiples of C, D. Whence, as A is to C, so shall B be * to D. Therefore, *if four Magnitudes of the same Kind are proportional, they shall also be alternately proportional.* Def. 8.

PROPOSITION XVII.

THEOREM.

If Magnitudes compounded are proportional, they shall also be proportional when divided.

LET the compounded Magnitudes AB, BE, CD, DF, be proportional; that is, let AB be to BE, as CD is to DF. I say, these Magnitudes divided are proportional, *viz.* as AE is to EB, so is CF to FD.

For, let GH, HK, LM, MN, be Equimultiples of AE, EB, CF, FD; and KX, NP, any Equimultiples of EB, FD.

* 1 of this.

Because GH is the same Multiple of AE, as HK is of EB; therefore GH* is the same Multiple of AE, as GK is of AB. But GH is the same Multiple of AE, as LM is of CF. Wherefore GK is the same Multiple of AB,

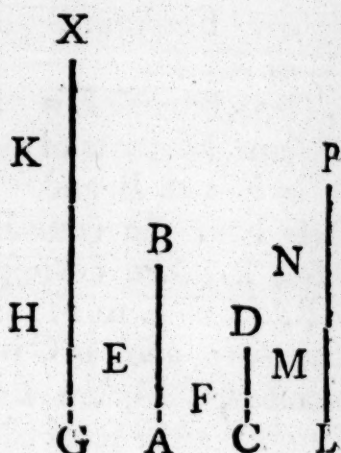
as LM is of CF. Again, because LM is the same Multiple of CF, as MN is of FD, LM will be

† 2 of this.

* the same Multiple of CF, as LM is of CD. Therefore GK is the same Multiple of AB, as LN is of CD. And so GK, LN, will be Equimultiples of AB, CD. Again, because HK is the same Multiple of EB, as MN is of FD; as likewise KX the same Multiple of EB, as NP is of FD; the compounded Magnitude HX is † also the same Multiple of EB, as MP is of FD. Wherefore, since it

‡ Def. 5.

is, as AB is to BE, so is CD to DF; and GK, LN, are Equimultiples of AB, CD; and also HX, MP, any Equimultiples of EB, FD; if GK exceeds HX, then LN will † exceed MP: And if GK be equal to HX, then LN will be equal to MP; if less, less. Now let GK exceed HX; then, if HK, which is common, be taken away, GH shall exceed KX. But when GK exceeds HX, then LN exceeds MP; therefore LN does exceed MP. If MN, which is common,



common, be taken away, then LM will exceed NP. And so, if GH exceeds KX, then LM will exceed NP. In like manner we demonstrate, if GH be equal to KX, that LM will be equal to NP; and if less, less. But GH, LM, are Equimultiples of AE, CF; and KX, NP, are any Equimultiples of EB, FD. Whence, * as AE is to EB, so CF to FD. * *Def. 5.* Therefore, *if Magnitudes compounded are proportional, they shall also be proportional when divided; which was to be demonstrated.*

PROPOSITION XVIII.

THEOREM.

If Magnitudes divided be proportional, the same also, being compounded, shall be proportional.

LET the divided proportional Magnitudes be AE, EB, CF, FD; that is, as AE is to EB, so is CF to FD. I say, they are also proportional when compounded; viz. as AB is to BE, so is CD to DF.

For, if AB be not to BE, as CD is to DF, AB shall be to BE, as CD is to a Magnitude, either greater or less than FD.

First, Let it be to a lesser, viz. to GD. Then, because AB is to BE, as CD is to DG, compounded Magnitudes are proportional; and consequently * they will be proportional when divided. Therefore AE is to

EB, as CG is to GD. But (by the *Hyp.*) as AE is to EB, so is CF to FD. Wherefore, also, as CG is to GD, so † is CF to FD. But the first CG is greater † *11 of this.* than the third CF; therefore the second DG shall be † greater than the fourth DF. But it is less, which is † *14 of this.* absurd. Therefore AB is not to BE, as CD is to DG. We demonstrate in the same manner, that AB to BE is not as CD to a greater than DF. Therefore AB to BE must necessarily be as CD is to DF. And so, if Magnitudes divided be proportional, they will also be proportional when compounded; *which was to be demonstrated.*



* 17 of this.

† 11 of this.

† 14 of this.

P R O-

PROPOSITION XIX.

THEOREM.

If the Whole be to the Whole, as a Part taken away is to a Part taken away; then shall the Residue be to the Residue, as the Whole is to the Whole.

LET the Whole AB be to the Whole CD, as the Part taken away AE is to the Part taken away CF. I say, the Residue EB is to the Residue FD, as the Whole AB is to the Whole CD.

For, because the Whole AB is to the Whole CD, * 16 of this. as AE is to CF; it shall be * alternately as AB is to AE, so is CD to CF. Then, because compounded Magnitudes, being proportional, will be

† 17 of this. † also proportional when divided; as BE is to EA, so is DF to FC: And again, it will be by Alternation, as BE to DF, so is EA to FC. But as EA to FC, so (by the Hyp.) is AB to CD. And therefore the Residue EB shall be to the Residue FD, as the Whole AB to the Whole CD. Wherefore, *if the Whole be to the Whole, as a Part taken away is to a Part taken away; then shall the Residue be to the Residue, as the Whole is to the Whole; which was to be demonstrated.*



Coroll. If four Magnitudes be proportional, they will be likewise conversly proportional. For let AB be to BE, as CD to DF; then (by Alternation) it shall be, as AB is to CD, so is BE to DF. Wherefore, since the Whole AB is to the Whole CD, as the Part taken away BE is to the Part taken away DF; the Residue AE to the Residue CF shall be as the Whole AB to the Whole CD. And again, (by Inversion and Alternation) as AB is to AE, so is CD to CF. Which is by converse Ratio.

The Demonstration of converse Ratio, laid down in this Corollary, is only particular. For Alternation (which is used herein) cannot be applied but when the four proportional Magnitudes are all of the same Kind, as will appear from the 4th and 17th Definitions of this Book. But converse Ratio may be used when the Terms

of

of the first Ratio are not of the same Kind with the Terms of the latter. Therefore, instead of that, it may not be improper to add this Demonstration following: If four Magnitudes are proportional, they will be so conversely: For let AB be to BE, as CD to DF. And then dividing, it is, as AE is to BE, so is CF to DF: And this inversely is, as BE is to AE, so is DF to CF; which by compounding becomes, as AB is to AE, so is CD to CF; which by the 17th Definition is converse Ratio: By S. Cunn.

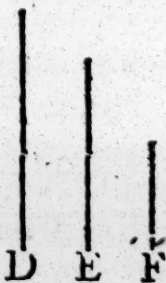
PROPOSITION XX.

THEOREM.

If there be three Magnitudes, and others equal to them in Number, which, being taken two and two in each Order, are in the same Ratio; and if the first Magnitude be greater than the third, then the fourth will be greater than the sixth: But if the first be equal to the third, then the fourth will be equal to the sixth; and if the first be less than the third, the fourth will be less than the sixth.

LET A, B, C, be three Magnitudes, and D, E, F, others equal to them in Number, taken two and two in each Order, are in the same Proportion, viz. let A be to B as D is to E, and B to C as E to F; and let the first Magnitude A be greater than the third C. I say, the fourth D is also greater than the sixth F. And if A be equal to C, D is equal to F. But if A be less than C, D is less than F.

For, because A is greater than C, and B is any other Magnitude; and since a greater Magnitude hath * a greater Proportion to the same Magnitude than a lesser hath, A will have a greater Proportion to B, than C to B. But as A is to B, so is D to E; and inversely, as C is to B, so is F to E. Therefore also D will have a greater Proportion to E, than F has to E. But of Magnitudes having Proportion to the same Magnitude, that which has the greater



* 8 of this.

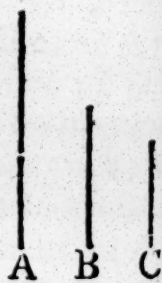
* 10 of this. Proportion is * the greater Magnitude. Therefore D is greater than F. In the same manner we demonstrate, if A be equal to C, then D will be also equal to F; and if A be less than C, then D will be less than F. Therefore, *if there be three Magnitudes, and others equal to them in Number, which being taken two and two in each Order, are in the same Ratio; if the first Magnitude be greater than the third, then the fourth will be greater than the sixth: But if the first be equal to the third, then the fourth will be equal to the sixth; and if the first be less than the third, the fourth will be less than the sixth; which was to be demonstrated.*

PROPOSITION XXI.

THEOREM.

If there be three Magnitudes, and others equal to them in Number, which, taken two and two, are in the same Proportion, and the Proportion be perturbate; if the first Magnitude be greater than the third, then the fourth will be greater than the sixth; but, if the first be equal to the third, then is the fourth equal to the sixth; if less, less.

LET three Magnitudes, A, B, C, be proportional; and others, D, E, F, equal to them in Number. Let their Analogy likewise be perturbate, viz. as A is to B, so is E to F; and as B is to C, so is D to E; if the first Magnitude A be greater than the third C, I say, the fourth D is also greater than the sixth F. And if A be equal to C, then D is equal to F; but if A be less than C, then D is less than F.

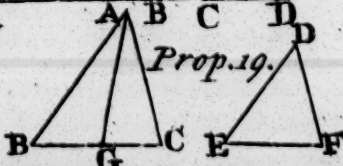
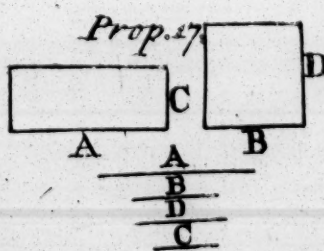
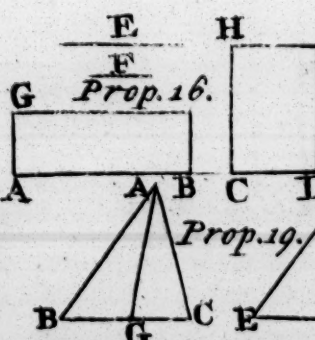
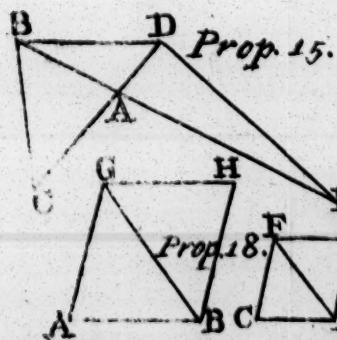
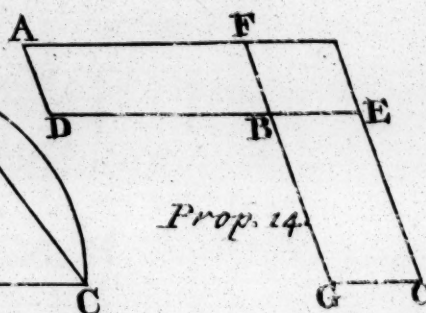
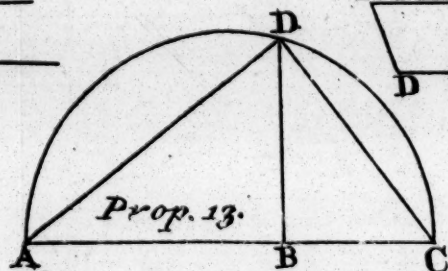
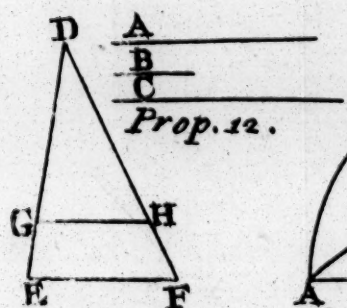
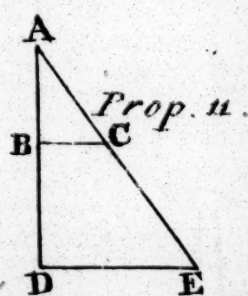
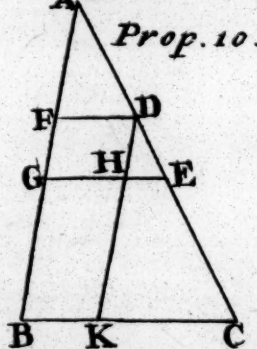
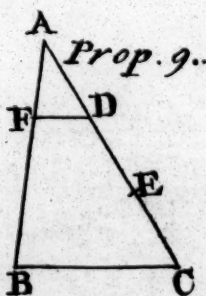
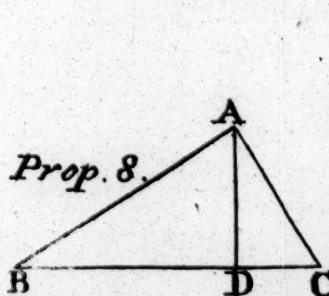
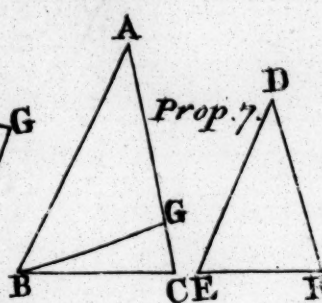
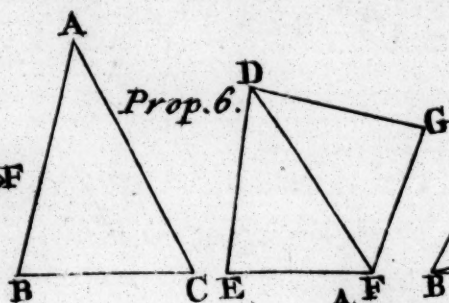
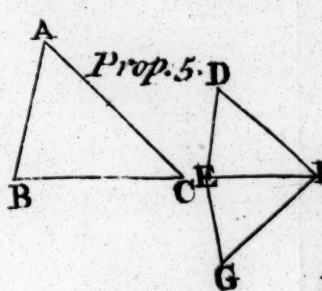
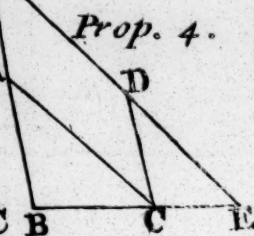
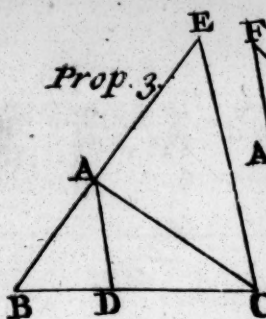
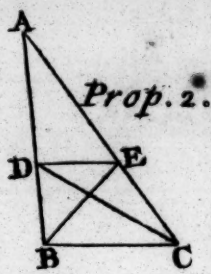
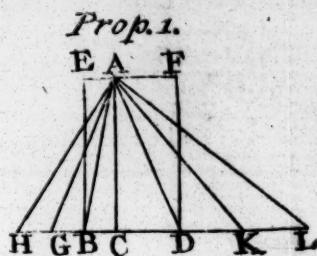
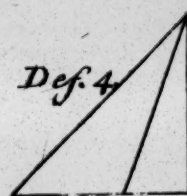
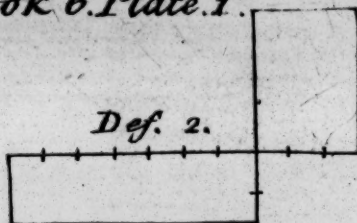


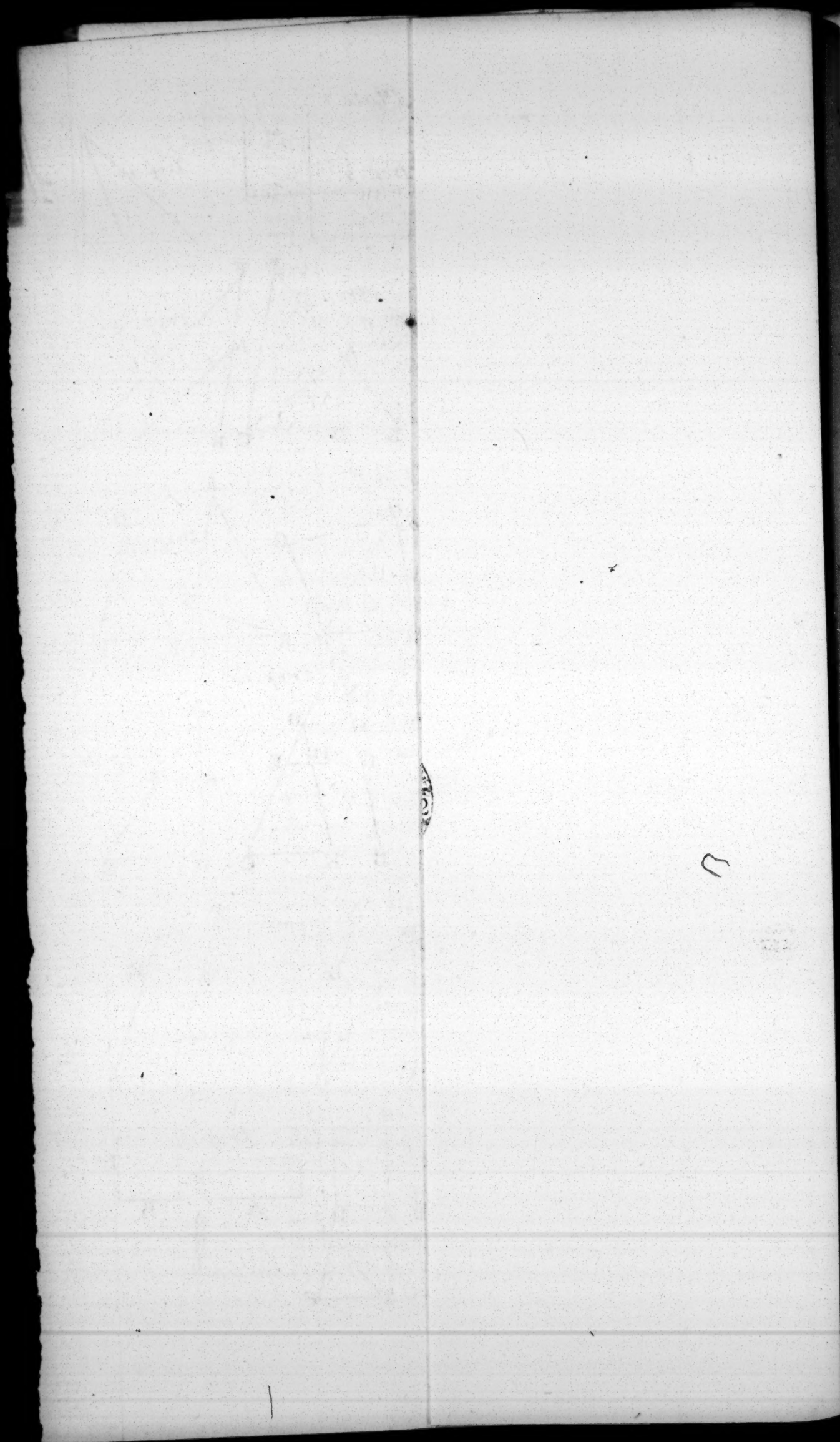
For, since A is greater than C, and B is some other Magnitude, A will have * a greater Proportion to B, than C has to B. But as A is to B, so is E to F; and inversely, as C is to B, so is E to D: Wherefore also E shall have a greater Proportion to F, than E to D. But that Magnitude to which the same Magnitude has a greater



† 10 of this. Proportion, is † the lesser Magnitude.

Therefore





Therefore F is less than D ; and so D shall be greater than F. After the same manner we demonstrate, if A be equal to C, D will be also equal to F ; and, if A be less than C, D will also be less than F. *If, therefore, there are three Magnitudes, and others equal to them in Number, which, taken two and two, are in the same Proportion, and the Proportion be perturbate ; if the first Magtitude be greater than the third, then the fourth will be greater than the sixth ; but, if the first be equal to the third, then is the fourth equal to the sixth ; if less, less ; which was to be demonstrated.*

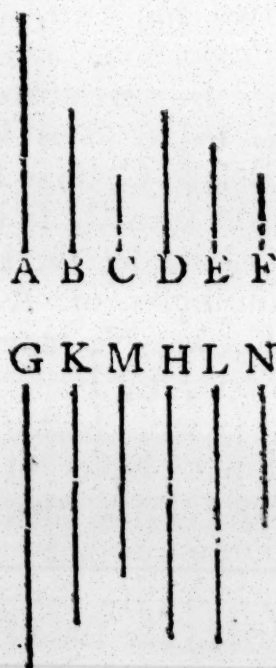
PROPOSITION XXII.

THEOREM.

If there be any Number of Magnitudes, and others equal to them in Number, which, taken two and two, are in the same Proportion ; then they shall be in the same Proportion by Equality.

LET there be any Number of Magnitudes, A, B, C, and others, D, E, F, equal to them in Number, which, taken two and two, are in the same Proportion, that is, as A is to B, so is D to E ; and as B is to C, so is E to F. I say, they are also proportional by Equality, viz. as A is to C, so is D to F.

For, let G, H, be Equimultiples of A, D ; and K, L, any Equimultiples of B, E ; and likewise M, N, any Equimultiples of C, F. Then, because A is to B, as D is to E ; and G, H, are Equimultiples of A, D ; and K, L, Equimultiples of B, E ; it shall be, * as G is to K, so is H to L. For the same Reason, also, it will be, as K is to M, so is L to N. And since there are three Magnitudes G, K, M, and others H, L, N, equal to them in



* 49. 1. 2.

Num-

Number, which, being taken two and two in each Order, are in the same Proportion ; if G exceeds M, * 20 of this. * H will exceed N ; if G be equal to M, then H shall be equal to N ; and if G be less than M, H shall be less than N. But, G, H, are Equimultiples of A, D ; and M, N, any other Equimultiples of C and F. Whence, as A is to C, so shall † D be to F. Therefore, if there be any Number of Magnitudes, and others equal to them in Number, which, taken two and two, are in the same Proportion ; then they shall be in the same Proportion by Equality ; which was to be demonstrated.

Def. 5.
of this.

PROPOSITION XXII.

THEOREM.

If there be three Magnitudes, and others equal to them in Number, which, taken two and two, are in the same Proportion ; and if their Analogy be perturbate, then shall they be also in the same Proportion by Equality.

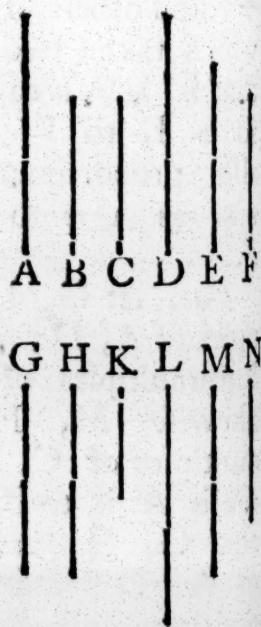
LET there be three Magnitudes A, B, C, and others equal to them in Number, D, E, F, which, taken two and two, are in the same Proportion, and their Analogy be perturbate, that is, as A is to B, so is E to F ; and as B is to C, so is D to E. I say, as A is to C, so is D to F.

For, let G, H, L, be Equimultiples of A, B, D, and K, M, N, any Equimultiples of C, E, F.

Then, because G, H, are Equimultiples of A and B, and since Parts have the same Proportion as their like Multiples, when taken corre-

* 15 of this. spondently, it shall be*, as A is to B, so is G to H ; and by the same Reason, as E is to F, so is M to N.

† 11 of this. But A is to B, as E is to F. Therefore, † as G is to H, so is M to N. Again, because B is to C, as D



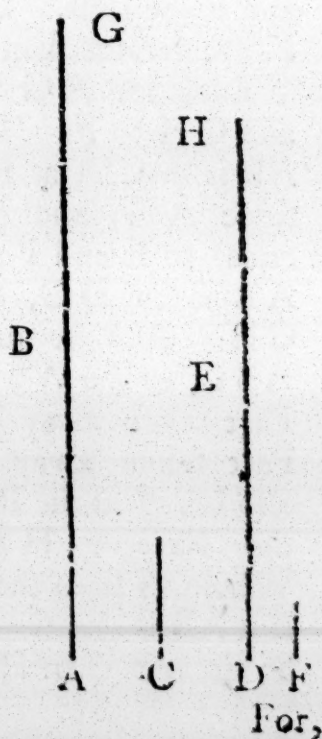
is to E; and H, L, are Equimultiples of B and D; as likewise K, M, any Equimultiples of C, E; it shall be, as H to K, so is L to M. But it has been also prov'd, that as G is to H, so is M to N. Therefore, because three Magnitudes, G, H, K, and others, L, M, N, equal to them in Number, which taken two and two are in the same Proportion, and their Analogy is perturbate; then if G exceeds K, also L* will exceed* 21 of this. N; and if G be equal to K, then L will be equal to N; and if G be less than K, L will likewise be less than N. But G, L, are Equimultiples of A, D; and K, N, Equimultiples of C, F. Therefore, as A is to C, so shall D be to F. Wherefore, *if there be three Magnitudes, and others equal to them in Number, which taken two and two are in the same Proportion; and if their Analogy be perturbate, then shall they be also in the same Proportion by Equality; which was to be demonstrated.*

PROPOSITION XXIV.

THEOREM.

If the first Magnitude has the same Proportion to the second, as the third to the fourth; and if the fifth has the same Proportion to the second, as the sixth has to the fourth, then shall the first, compounded with the fifth, have the same Proportion to the second, as the third, compounded with the sixth, has to the fourth.

LET the first Magnitude AB have the same Proportion to the second C, as the third DE has to the fourth F. Let also the fifth BG have the same Proportion to the second C, as the sixth EH has to the fourth F. I say, AG the first compounded with the fifth, has the same Proportion to the second C, as DH the third compounded with the sixth, has to the fourth F.



For, because BG is to C, as EH is to F, it shall be (inversly) as C is to BG, so is F to EH. Then, since AB is to C, as DE is to F; and as C is to BG, so is F to EH; it shall be * by Equality, as AB is to BG, so is DE to EH. And because Magnitudes; being divided, are proportional, they shall also be † proportional when compounded. Therefore, as AG is to GB, so is DH to HE: But as GB is ‡ to C, so also is HE to F. Wherefore, by Equality*, it shall be, as AG is to C, so is DH to F. Therefore, *if the first Magnitude has the same Proportion to the second, as the third to the fourth; and if the fifth has the same Proportion to the second, as the sixth has to the fourth; then shall the first, compounded with the fifth, have the same Proportion to the second, as the third, compounded with the sixth, has to the fourth; which was to be demonstrated.*

PROPOSITION XXV.

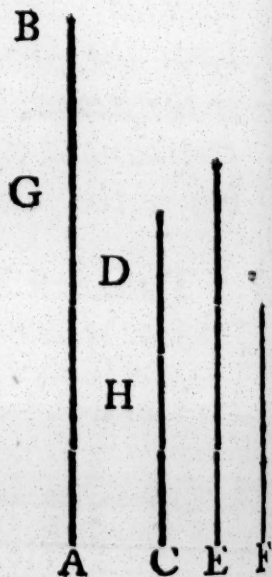
THEOREM.

If four Magnitudes be proportional, the greatest and the least of them will be greater than the other two.

LET four Magnitudes, AB, CD, E, F, be proportional, whereof AB is to CD, as E is to F; let AB be the greatest of them, and F the least. I say, AB, and F, are greater than CD, and E.

For, let AG be equal to E, and CH to F. Then, because AB is to CD, as E is to F; and since AG, and CH, are each equal to E and F; it shall be as AB is to DC, so is AG to CH. And because the Whole AB is, to the Whole CD, as the Part taken away AG, is to the Part taken away CH; it shall also be, * as the Residue GB to the Residue HD, so is the Whole AB to the Whole CD.

But AB is greater than CD; therefore, also, GB shall be greater than HD. And since AG is equal to E, and



and CH to F, AG and F will be equal to CH and E. But if equal Things are added to unequal Things, the Wholes shall be unequal. Therefore GB, HD, being unequal, for GB is the greater; if AG, and F, are added to GB; and CH, and E, to HD; AB and F will necessarily be greater than CD and E. Wherefore, *if four Magnitudes be proportional, the greatest and the least of them will be greater than the other two*; which was to be demonstrated.

The END of the FIFTH BOOK.

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GB shall
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and

E U C L I D's ELEMENTS.

B O O K VI.

DEFINITIONS.

- I. **S**IMILAR right-lined Figures are such as have each of their several Angles equal to one another, and the Sides about the equal Angles proportional to each other.
- II. Figures are said to be reciprocal, when the antecedent and consequent Terms of the Ratios are in each Figure.
- III. A right Line is said to be cut into mean and extreme Proportion, when the Whole is to the greater Segment, as the greater Segment is to the lesser.
- IV. The Altitude of any Figure is a perpendicular Line drawn from the Top, or Vertex, to the Base.
- V. A Ratio is said to be compounded of Ratios, when the Quantities of the Ratios, being multiplied into one another, do produce a Ratio.

PROPOSITION I.

THEOREM.

Triangles and Parallelograms, that have the same Altitude, are to each other as their Bases.

LET the Triangles ABC , ACD , and the Parallelograms EC , CF , have the same Altitude, *viz.* the Perpendicular drawn from the Point A to BD . I say, as the Base BC is to the Base CD , so is the Triangle ABC to the Triangle ACD ; and so is the Parallelogram EC to the Parallelogram CF .

For, produce BD both ways to the Points H and L , and take GB , GH , any Number of Times equal to the Base BC ; and DK , KL , any Number of Times equal to the Base CD ; and join AG , AH , AK , AL .

Then, because CB , BG , GH , are equal to one another, the Triangles AHG , AGB , ABC , also, will be * equal to one another: Therefore the same * ^{38. 1.} Multiple that the Base HC is of BC , shall the Triangle AHC be of the Triangle ABC . By the same Reason, the same Multiple that the Base LC is of the Base CD , shall the Triangle ALC be of the Triangle ACD . And if HC be equal to the Base CL , the Triangle AHC is also * equal to the Triangle ALC : And if the Base HC exceeds the Base CL , then the Triangle AHC will exceed the Triangle ALC . And if HC be less, then the Triangle AHC will be less. Therefore since there are four Magnitudes, *viz.* the two Bases BC , CD , and the two Triangles ABC , ACD ; and since the Base HC , and the Triangle AHC , are Equimultiples of the Base BC , and the Triangle ABC : And the Base CL , and the Triangle ALC , are Equimultiples of the Base CD , and the Triangle ADC : And it has been proved, that if the Base HC exceeds the Base CL , the Triangle AHC will exceed the Triangle ALC ; and if equal, equal; if less, less: Then, as the Base BC is to the Base CD , so † is the Triangle † *Def. 5.* ABC to the Triangle ACD .

L 2

And

† 41. 1.

† 11. 5.

And because the Parallelogram EC is † double to the Triangle ABC ; and the Parallelogram FC double † to the Triangle ACD ; and Parts have the same Proportion as their like Multiples: Therefore, as the Triangle ABC is to the Triangle ACD , so is the Parallelogram EC to the Parallelogram CF . And so, since it has been proved, that the Base BC is to the Base CD , as the Triangle ABC is to the Triangle ACD ; and the Triangle ABC is to the Triangle ACD , as the Parallelogram EC is to the Parallelogram CF ; it shall be, † as the Base BC is to the Base CD , so is the Parallelogram EC to the Parallelogram FC . Wherefore, *Triangles and Parallelograms, that have the same Altitude, are to each other as their Bases*; which was to be demonstrated.

PROPOSITION II.

THEOREM.

If a right Line be drawn parallel to one of the Sides of a Triangle, it shall cut the Sides of the Triangle proportionally; and if the Sides of the Triangle be cut proportionally, then a right Line, joining the Points of Section, shall be parallel to the other Side of the Triangle.

LET DE be drawn parallel to BC , a Side of the Triangle ABC . I say, DB is to DA , as CE is to EA .

For, let BE , CD , be joined.

* 37. 1.

† 7. 5.

Then the Triangle BDE is * equal to the Triangle CDE ; for they stand upon the same Base DE , and are between the same Parallels DE and BC ; and ADE is some other Triangle. But equal Magnitudes have † the same Proportion to one and the same Magnitude. Therefore, as the Triangle BDE is to the Triangle ADE , so is the Triangle CDE to the Triangle ADE .

But as the Triangle BDE is to the Triangle ADE , so † is BD to DA ; for since they have the same Altitude, viz. a Perpendicular drawn from the Point E to AB , they are to each other as their Bases. And, for the same Reason, as the Triangle CDE is to the Triangle ADE , so is CE to EA .

† 1 of this.

to the Triangle ADE, so is CE to EA; And therefore, as BD is to DA, so * is CE to EA.

* 11. 1.

And if the Sides AB, AC, of the Triangle ABC, be cut proportionally, that is, so that BD be to DA, as CE is to EA; and if DE be joined; I say, DE is parallel to BC.

For, the same Construction remaining, because BD is to DA, as CE is to EA; and BD is † to DA, as † 1 of this. the Triangle BDE is to the Triangle ADE; and CE is to EA, as the Triangle CDE is to the Triangle ADE; it shall be, as the Triangle BDE is to the Triangle ADE, so is * the Triangle CDE to the Triangle ADE. And since the Triangles BDE, CDE, have the same Proportion to the Triangle ADE, the Triangle BDE shall be † equal to the Triangle CDE; † 9. 5. and they have the same Base DE: But equal Triangles, being upon the same Base, † are between the † 39. 1. same Parallels; therefore DE is parallel to BC. Wherefore, if a right Line be drawn parallel to one of the Sides of a Triangle, it shall cut the Sides of the Triangle proportionally; and if the Sides of the Triangle be cut proportionally, then a right Line, joining the Points of Section, shall be parallel to the other Side of the Triangle; which was to be demonstrated.

PROPOSITION III.

THEOREM.

If one Angle of a Triangle be bisected, and the right Line, that bisects the Angle, cuts the Base also; then the Segments of the Base will have the same Proportion as the other Sides of the Triangle. And if the Segments of the Base have the same Proportion that the other Sides of the Triangle have; then a right Line, drawn from the Vertex to the Point of Section of the Base, will bisect the Angle of the Triangle.

LET there be a Triangle ABC, and let its Angle BAC be * bisected by the right Line AD. I * 9. 1. say, as BD is to DC, so is BA to AC.

L 3

For,

* 31. 1.

For, thro' C draw * CE parallel to DA, and produce BA, till it meets CE in the Point E.

† 29. 1.

Then, because the right Line AC falls on the Parallels AD, EC, the Angle ACE will be † equal to the Angle CAD: But the Angle CAD (by the Hypothesis) is equal to the Angle BAD. Therefore the Angle BAD will be equal to the Angle ACE. Again, because the right Line BAE falls on the Parallels AD, EC, the outward Angle BAD is † equal to the inward Angle AEC; but the Angle ACE has been proved equal to the Angle BAD: Therefore ACE shall be equal to AEC; and so the

† 6. 1.

Side AE is equal † to the Side AC. And because the Line AD is drawn parallel to CE, the Side of the

* 2 of this.

Triangle BCE, it shall be, * as BD is to DC, so is BA to AE; but AE is equal to AC. Therefore, as

† 7. 5.

BD is to DC, so is † BA to AC.

And if BD be to DC, as BA is to AC; and the right Line AD be joined; then, I say, the Angle BAC is bisected by the right Line AD.

† 2 of this.

For the same Construction remaining, because BD is to DC, as BA is to AC; and as BD is to DC, so is † BA to AE; for AD is drawn parallel to one Side EC of the Triangle BCE; it shall be, as BA is to AC, so is BA to AE. Therefore AC is equal to AE; and, accordingly, the Angle AEC is equal to the Angle ECA: But the Angle AEC is equal

* 25. 1.

* to the outward Angle BAD; and the Angle ACE equal to the alternate Angle CAD. Wherefore the Angle BAD is also equal to the Angle CAD; and so the Angle BAC is bisected by the right Line AD. Therefore, *if the Angle of a Triangle be bisected, and the right Line, that bisects the Angle, cuts the Base also; then the Segments of the Base will have the same Proportion as the other Sides of the Triangle. And if the Segments of the Base have the same Proportion that the other Sides of the Triangle have; then a right Line, drawn from the Vertex to the Point of Section of the Base, will bisect the Angle of the Triangle; which was to be demonstrated.*

PROPOSITION IV.

THEOREM.

The Sides about the equal Angles of equiangular Triangles are proportional; and the Sides which are subtended under the equal Angles, are homologous, or of like Ratio.

LET ABC , DCE , be equiangular Triangles, having the Angle ABC equal to the Angle DCE , the Angle ACB equal to the Angle DEC , and the Angle BAC equal to the Angle CDE . I say, the Sides that are about the equal Angles of the Triangles ABC , DCE , are proportional; and the Sides that are subtended under the equal Angles, are homologous, or of like Ratio.

Set the Side BC in the same right Line with the Side CE ; and because the Angles ABC , ACB , are * less than two right Angles, and the Angle ACB * 17. 1. is equal to the Angle DEC , the Angles ABC , DEC , are less than two right Angles. And so BA , ED , produced, will meet † each other; let them be pro-† *Ax.* 12. duced, and meet in the Point F . Then, because the Angle DCE is equal to the Angle ABC , BF shall be † parallel to DC . Again, because the Angle ACB † 28. 1. is equal to the Angle DEC , the Side AC will be † parallel to the Side FE ; therefore $FACD$ is a Parallelogram, and consequently FA is * equal to DC , * 34. 1. and AC to FD ; and because AC is drawn parallel to FE , the Side of the Triangle FBE , it shall † be, † 2 of this. as BA is to AF , so is BC to CE ; and (by Alternation) as BA is to BC , so is CD to CE . Again; because CD is parallel to BF , it shall be, † as BC is to CE , so is FD to DE ; but FD is equal to AC . Therefore, as BC is to CE , so is † AC to DE : And † 7. 5. so (by Alternation) as BC is to CA , so is CE to ED . Wherefore, because it is demonstrated, that AB is to BC , as DC is to CE ; and as BC is to CA , so is CE to ED ; it shall be, * by Equality, as BA is to * 21. 1. AC , so is CD to DE . Therefore, *the Sides about the equal Angles of equiangular Triangles are proportional; and the Sides which are subtended under the*

equal Angles, are homologous, or of like Ratio; which was to be demonstrated.

PROPOSITION V.

THEOREM.

If the Sides of two Triangles are proportional, the Triangles shall be equiangular; and their Angles, under which the homologous Sides are subtended, are equal.

LET there be two Triangles, ABC, DEF, having their Sides proportional, that is, let AB be to BC, as DE is to EF; and as BC to CA, so is EF to FD. And, also, as BA to CA, so ED to DF. I say, the Triangle ABC is equiangular to the Triangle DEF; and the Angles equal, under which the homologous Sides are subtended, *viz.* the Angle ABC equal to the Angle DEF; and the Angle BCA equal to the Angle EFD; and the Angle BAC equal to the Angle EDF.

For, at the Points E and F, with the Line EF, make
 * 23. 1. * the Angle FEG equal to the Angle ABC; and the Angle EFG equal to the Angle BCA: Then the remaining Angle BAC is † equal to the remaining Angle EGF.
 † Cor. 32. 1.

And so the Triangle ABC is equiangular to the Triangle EGF; and, consequently, the Sides that are subtended under the equal Angles, are proportional.
 † 4 of this. Therefore, as AB is to BC, so is † GE to EF; but as AB is to BC, so is DE to EF: Therefore, as DE is to EF, so is * GE to EF. And since DE, EG, have the same Proportion to EF, DE shall be † equal to EG. For the same Reason, DF is equal to FG; but EF is common. Then, because the two Sides DE, EF, are equal to the two Sides GE, EF, and the Base DF is equal to the Base FG, the Angle DEF is † equal to the Angle GEF; and the Triangle DEF equal to the Triangle GEF; and the other Angles of the one equal to the other Angles of the other, which are subtended by the equal Sides. Therefore the Angle DEF is equal to the Angle GEF, and the Angle EDF equal to the Angle EGF.
 * 11. 5.
 † 9. 5.
 † 8. 1.

EGF. And because the Angle DEF is equal to the Angle GEF; and the Angle GEF equal to the Angle ABC; therefore the Angle ABC shall be also equal to the Angle FED: For the same Reason, the Angle ACB shall be equal to the Angle DFE; as also the Angle A equal to the Angle D; therefore the Triangle ABC will be equiangular to the Triangle DEF. Wherefore, *if the Sides of two Triangles are proportional, the Triangles shall be equiangular; and their Angles, under which the homologous Sides are subtended, are equal; which was to be demonstrated.*

PROPOSITION VI.

THEOREM.

If two Triangles have one Angle of the one equal to one Angle of the other; and if the Sides about the equal Angles be proportional; then the Triangles are equiangular, and have those Angles equal, under which are subtended the homologous Sides.

LET there be two Triangles ABC, DEF, having one Angle BAC of the one equal to the Angle EDF of the other; and let the Sides about the equal Angles be proportional, viz. let AB be to AC, as ED is to DF. I say, the Triangle ABC is equiangular to the Triangle DEF; and the Angle ABC equal to the Angle DEF; and the Angle ACB equal to the Angle DFE.

For, at the Points D and F, with the right Line DF, make * the Angle FDG equal to either of the * 23. 1. Angles BAC, EDF; and the Angle DFG equal to the Angle ACB.

Then the other Angle B is † equal to the other † Cor. 32. 1. Angle G; and so the Triangle ABC is equiangular to the Triangle DGF; and, consequently, as BA is to AC, so is ‡ GD to DF: But (by the Hyp.) as ‡ 4 of this. BA is to AC, so is ED to DF. Therefore, as ED is * to DF, so is GD to DF; whence ED is † equal * 11. 5. to DG, and DF is common; therefore the two Sides † 9. 5. ED, DF, are equal to the two Sides GD, DF; and the Angle EDF, equal to the Angle GDF:

Con-

4.1.

† By Hyp.

Consequently the Base EF is * equal to the Base FG , and the Triangle DEF equal to the Triangle GDF ; and the other Angles of the one equal to the other Angles of the other, each to each; under which the equal Sides are subtended. Therefore the Angle DFG is equal to the Angle DFE , and the Angle G equal to the Angle E ; but the Angle DFG is equal to the Angle ACB : Wherefore the Angle ACB is equal to the Angle DFE ; and the Angle BAC is † also equal to the Angle EDF : Therefore the other Angle at B is equal to the other Angle at E ; and so the Triangle ABC is equiangular to the Triangle DEF . Therefore, if two Triangles have one Angle of the one equal to one Angle of the other; and if the Sides about the equal Angles be proportional; then the Triangles are equiangular, and have those Angles equal, under which are subtended the homologous Sides; which was to be demonstrated.

PROPOSITION VII.

THEOREM.

If there are two Triangles, having one Angle of the one equal to one Angle of the other, and the Sides about other Angles proportional; and if the remaining third Angles are either both less, or both not less, than right Angles; then shall the Triangles be equiangular, and have those Angles equal, about which are the proportional Sides.

LET two Triangles ABC , DEF , have one Angle of the one equal to one Angle of the other, viz. the Angle BAC equal to the Angle EDF ; and let the Sides about the other Angles ABC , DEF , be proportional; viz. as DE is to EF , so let AB be to BC ; and let the other Angles at C and F be both less, or both not less, than right Angles. I say, the Triangle ABC is equiangular to the Triangle DEF ; and the Angle ABC is equal to the Angle DEF ; as also the other Angle at C equal to the other Angle at F .

For, if the Angle ABC be not equal to the Angle DEF , one of them will be the greater, which let be ABC .

ABC. Then at the Point B, with the right Line AB, make * the Angle ABG equal to the Angle DEF.

Now, because the Angle A is equal to the Angle D, and the Angle ABG equal to the Angle DEF; the remaining Angle AGB is † equal to the remaining Angle DFE: And therefore the Triangle ABG is equiangular to the Triangle DEF; and so, as AB is to BG, so is † DE to EF; but as DE is to EF, † 4 of this. so is * AB to BC. Therefore, as AB is to BC, so is * By Hyp. AB to BG; and since AB has the same Proportion to BC that it has to BG, BC shall be † equal to BG; and consequently the Angle at C equal to the Angle BGC. Wherefore each of the Angles BCG, or BGC, is less than a right Angle; and consequently AGB is greater than a right Angle. But the Angle AGB has been proved equal to the Angle at F; therefore the Angle at F is greater than a right Angle: But (by the Hyp.) it is not greater, since C is not greater than a right Angle, which is absurd. Wherefore the Angle ABC is not unequal to the Angle DEF; and so it must be equal to the same; but the Angle at A is equal to that at D; wherefore the Angle remaining at C is equal to the remaining Angle at F; and consequently the Triangle ABC is equiangular to the Triangle DEF. Therefore, if there are two Triangles having one Angle of the one equal to one Angle of the other, and the Sides about other Angles proportional; and if the remaining third Angles are either both less, or both not less, than right Angles; then shall the Triangles be equiangular, and have those Angles equal, about which are the proportional Sides; which was to be demonstrated.

PROPOSITION VIII.

THEOREM.

If a Perpendicular be drawn, in a right-lined Triangle, from the right Angle to the Base, then the Triangles on each Side of the Perpendicular are similar both to the Whole, and also to one another.

LET ABC be a right-angled Triangle, whose right Angle is BAC ; and let the Perpendicular AD be drawn from the Point A to the Base BC . I say, the Triangles ABD , ADC , are similar to one another, and to the whole Triangle ABC .

For, because the Angle BAC is equal to the Angle ADB , for each of them is a right Angle; and the Angle at B is common to the two Triangles ABC , ABD ; the remaining Angle ACB shall be * equal to the remaining Angle BAD . Therefore the Triangle ABC is equiangular to the Triangle ABD ; and so, as † BC , which subtends the right Angle of the Triangle ABC , is to BA , subtending the right Angle of the Triangle ABD , so is AB , subtending the Angle C of the Triangle ABC , to DB , subtending an Angle equal to the Angle C , viz. the Angle BAD , of the Triangle ABD . And so, moreover, is AC to AD , subtending the Angle B , which is common to the two Triangles. Therefore the Triangle ABC is ‡ equiangular to the Triangle ABD ; and the Sides about the equal Angles are proportional. Wherefore the Triangle ABC is ‡ similar to the Triangle ABD . By the same way we demonstrate, that the Triangle ADC is also similar to the Triangle ABC . Wherefore each of the Triangles ABD , ADC , is similar to the whole Triangle.

I say, the said Triangles are also similar to one another.

For, because the right Angle BDA is equal to the right Angle ADC , and the Angle BAD has been proved equal to the Angle C ; it follows, that the remaining Angle at B shall be equal to the remaining Angle DAC . And so the Triangle ABD is equiangular

* Cor. 32. 1.

† 4 of this.

‡ Def. 1. of this.

angular to the Triangle ADC. Wherefore, as $\dagger$ BD, $\dagger$ 4 of this, subtending the Angle BAD of the Triangle ABD, is to DA, subtending the Angle at C of the Triangle ADC, which is equal to the Angle BAD; so is AD, subtending the Angle B of the Triangle ABD, to DC, subtending the Angle DAC, equal to the Angle B. And, moreover, so is BA to AC, subtending the right Angles at D; and, consequently, the Triangle ABD is similar to the Triangle ADC. Wherefore, *if a Perpendicular be drawn, in a right-angled Triangle, from the right Angle to the Base, then the Triangles on each Side of the Perpendicular are similar both to the Whole, and also to one another; which was to be demonstrated.*

Coroll. From hence it is manifest, that the Perpendicular drawn in a right-angled Triangle from the right Angle to the Base, is a mean Proportional between the Segments of the Base. Moreover, either of the Sides containing the right Angle, is a mean Proportional between the whole Base, and that Segment thereof which is next to the Side.

PROPOSITION IX.

PROBLEM.

To cut off any Part required from a given right Line.

LET AB be a right Line given, from which must be cut off any required Part, suppose a third.

Draw any right Line AC from the Point A, making an Angle at Pleasure with the Line AB. Assume any Part D in the Line AC; make * DE, EC, * 3. 1. each equal to AD; join BC, and draw $\dagger$ DF thro' $\dagger$ 3 1. 1. D, parallel to BC.

Then, because FD is drawn parallel to the Side BC of the Triangle ABC, it shall be, $\dagger$ as CD is $\dagger$ 2 of this, to DA, so is BF to FA. But CD is double to DA. Therefore BF shall be double to FA; and so BA is triple to AF. Wherefore, there is cut off AF, a third Part required, of the given right Line AB; *which was to be done.*

P R O-

PROPOSITION X.

PROBLEM.

To divide a given undivided right Line, as another given right Line is divided.

LET AB be a given undivided right Line, and AC a divided Line. It is required to divide AB, as AC is divided.

Let AC be divided in the Points D and E, and so placed, as to contain any Angle with AB. Join the Points C and B; thro' D and E let DF, EG, be drawn * parallel to BC; and through D draw DHK, parallel to AB.

Then FH, HB, are each of them Parallelograms; and so DH is † equal to FG; and HK to GB. And because HE is drawn parallel to the Side KC, of the Triangle DKC, it shall be ‡ as CE is to ED, so is KH to HD. But KH is equal to BG, and HD to GF. Therefore, as CE is to ED, so is BG to GF. Again, because FD is drawn parallel to the Side EG of the Triangle AGE, as ED is to DA, so shall † GF be to FA. But it has been proved, that CE is to ED, as BG is to GF. Therefore, as CE is to ED, so is BG to GF; and as ED is to DA, so is GF to FA. Wherefore the given undivided Line AB is divided as the given Line AC is; *which was to be done.*

PROPOSITION XI.

PROBLEM.

Two right Lines being given, to find a third Proportional to them.

LET AB, AC, be two given right Lines, so placed, as to make any Angle with each other. It is required to find a third Proportional to AB, AC.

Produce AB, AC, to the Points D and E; make BD equal to AC; join the Points B, C; and draw * the right Line DE thro' D parallel to BC.

I

Then,

Then, because BC is drawn parallel to the Side DE of the Triangle ADE, it shall be, * as AB is to BD, so is AC to CE. But BD is equal to AC. Hence, as AB is to AC, so is AC to CE. Therefore a third Proportional CE is found to two given right Lines AB, AC; *which was to be done.* † 2 of this.

PROPOSITION XII.

PROBLEM.

Three right Lines being given, to find a fourth Proportional to them.

LET A, B, C, be three right Lines given. It is required to find a fourth Proportional to them.

Let DE and DF be two right Lines, making any Angle EDF with each other. Now make DG equal to A, GE equal to B, DH equal to C; and draw the Line GH, as also † EF thro' E, parallel to GH. † 31. 1.

Then, because GH is drawn parallel to EF, the Side of the Triangle DEF, it shall be, as DG is to GE, so is DH to HF. But DG is equal to A, GE to B, and DH to C. Consequently, as A is to B, so is C to HF. Therefore the right Line HF, a fourth Proportional to the three given right Lines A, B, C, is found; *which was to be done.*

PROPOSITION XIII.

PROBLEM.

To find a mean Proportional between two given right Lines.

LET the two given right Lines be AB, BC. It is required to find a mean Proportional between them. Place AB, BC, in a direct Line; and on the Whole AC describe the Semicircle ADC, and * draw * 11. 1. BD at right Angles to AC from the Point B, and let AD, DC, be joined.

Then, because the Angle ADC, in a Semicircle, is † a right Angle; and since the Perpendicular DB is † 31. 3. drawn from the right Angle to the Base; therefore

DB

* Cor. 8.
of this.

DB is * a mean Proportional between the Segments of the Base AB, BC. Wherefore a mean Proportional between the two given Lines AB, BC, is found; which was to be done.

PROPOSITION XIV.

THEOREM.

Equal Parallelograms, having one Angle of the one equal to one Angle of the other, have the Sides about the equal Angles reciprocal; and those Parallelograms that have one Angle of the one equal to one Angle of the other, and the Sides that are about the equal Angles reciprocal, are equal between themselves.

* 14. 1.

LET AB, BC, be equal Parallelograms, having the Angles at B equal; and let the Sides DB, BE, be in one strait Line; then also will * the Sides FB, BG, be in one strait Line. I say, the Sides of the Parallelograms AB, BC, that are about the equal Angles, are reciprocal; that is, as DB is to BE, so is GB to BF.

For, let the Parallelogram FE be completed.

† 7. 5.

† 1 of this.

Then, because the Parallelogram AB is equal to the Parallelogram BC, and FE is some other Parallelogram; it shall be, as AB is to FE, so is † BC to FE; but as AB is to FE, so is † DB to BE; and as BC is to FE, so is GB to BF. Therefore, as DB is to BE, so is GB to BF. Wherefore the Sides of the Parallelograms AB, BC, that are about the equal Angles, are reciprocally proportional.

And if the Sides that are about the equal Angles are reciprocally proportional, viz. if BD be to BE, as GB is to BF; I say, the Parallelogram AB is equal to the Parallelogram BC.

For, since DB is to BE, as GB is to BF; and DB to BE, as the Parallelogram AB † to the Parallelogram FE; and GB † to BF, as the Parallelogram BC to the Parallelogram FE; it shall be, as AB is to FE, so is BC to FE. Therefore the Parallelogram AB is equal to the Parallelogram BC. And so, *equal Parallelograms, having one Angle of the one equal to one Angle of the other, have the Sides about the equal Angles*

gles reciprocal; and those Parallelograms that have one Angle of the one equal to one Angle of the other, and the Sides that are about the equal Angle reciprocal, are equal between themselves; which was to be demonstrated.

PROPOSITION XV.

THEOREM.

Equal Triangles, having one Angle of the one equal to one Angle of the other, have their Sides about the equal Angles reciprocal; and those Triangles that have one Angle of the one equal to one Angle of the other, and have also the Sides about the equal Angles reciprocal, are equal between themselves.

LET the equal Triangles ABC , ADE , have one Angle of the one equal to one Angle of the other, viz. the Angle BAC equal to the Angle DAE . I say, the Sides about the equal Angles are reciprocal; that is, as CA is to AD , so is EA to AB .

For, place CA and AD in one strait Line; then EA and AB shall be * also in one strait Line; and let * 14. 1. BD be joined. Then, because the Triangle ABC is equal to the Triangle ADE , and ABD is some other Triangle, the Triangle CAB shall be † to the Tri- † 7. 5. angle BAD , as the Triangle ADE is to the Triangle BAD . But, as the Triangle CAB is to the Triangle BAD , so is CA ‡ to AD ; and as the Tri- † 1 of this. angle EAD is to the Triangle BAD , so ‡ is EA to AB . Therefore, as CA is to AD , so is EA to AB . Wherefore the Sides of the Triangles ABC , ADE , about the equal Angles, are reciprocal.

And, if the Sides about the equal Angles of the Triangles ABC , ADE , be reciprocal, viz. if CA be to AD , as EA is to AB ; I say, the Triangle ABC is equal to the Triangle ADE .

For, again, let BD be joined. Then, because CA is to AD , as EA is to AB ; and CA to AD †, as the Triangle ABC to the Triangle BAD ; and EA to AB †, as the Triangle EAD to the Triangle BAD ; therefore, as the Triangle ABC is to the Triangle BAD , so shall the Triangle EAD be to the Triangle

M

BAD.

BAD. Whence the Triangles ABC, ADE, have the same Proportion to the Triangle BAD; and so the Triangle ABC is equal to the Triangle ADE. Therefore, *equal Triangles, having one Angle of the one equal to one Angle of the other, have their Sides about the equal Angles reciprocal; and those Triangles that have one Angle of the one equal to one Angle of the other, and have also the Sides about the equal Angles reciprocal, are equal between themselves; which was to be demonstrated.*

PROPOSITION XVI.

THEOREM.

If four right Lines be proportional, the Rectangle contained under the Extremes is equal to the Rectangle contained under the Means; and if the Rectangle contained under the Extremes be equal to the Rectangle contained under the Means, then are the four right Lines proportional.

LET four right Lines AB, CD, E, F, be proportional, so that AB be to CD, as E is to F. I say, the Rectangle contained under the right Lines AB and F, is equal to the Rectangle contained under the right Lines CD and E.

For, draw AG, CH, from the Points A, C, at right Angles to AB and CD; and make AG equal to F, and CH equal to E; and let the Parallelograms BG, DH, be completed.

Then, because AB is to CD, as E is to F; and since CH is equal to E, and AG to F; it shall be, as AB is to CD, so is CH to AG. Therefore, the Sides that are about the equal Angles of the Parallelograms BG, DH, are reciprocal; and since those Parallelograms are equal * that have the Sides about the equal Angles reciprocal; therefore the Parallelogram BG is equal to the Parallelogram DH. But the Parallelogram BG is equal to that contained under AB and F; for AG is equal to F, and the Parallelogram DH equal to that contained under CD and E, since CH is equal to E. Therefore the Rectangle contained under AB and F, is equal to that contained under CD and E.

And

* 14 of this.

And if the Rectangle contained under AB and F , be equal to the Rectangle contained under CD and E ; I say, the four right Lines are Proportionals, *viz.* as AB is to CD , so is E to F .

For, the same Construction remaining, the Rectangle contained under AB and F , is equal to that contained under CD and E ; but the Rectangle contained under AB and F , is the Rectangle BG ; for AG is equal to F ; and the Rectangle contained under CD and E , is the Rectangle DH ; for CH is equal to E . Therefore the Parallelogram BG shall be equal to the Parallelogram DH , and they are equiangular; but the Sides of equal and equiangular Parallelograms, which are about the equal Angles, are * reciprocal. Wherefore, as AB is to CD , so is CH * 14 of this, to AG ; but CH is equal to E , and AG to F ; therefore, as AB is to CD , so is E to F . Wherefore, *if four right Lines be proportional, the Rectangle contained under the Extremes, is equal to the Rectangle contained under the Means; and if the Rectangle contained under the Extremes be equal to the Rectangle contained under the Means, then are the four right Lines proportional;* which was to be demonstrated.

PROPOSITION XVII.

THEOREM.

If three right Lines be proportional, the Rectangle contained under the Extremes is equal to the Square of the Mean; and if the Rectangle under the Extremes be equal to the Square of the Mean, then the three right Lines are proportional.

LET there be three right Lines, A , B , C , proportional; and let A be to B , as B is to C . I say, the Rectangle contained under A and C , is equal to the Square of B .

For, make D equal to B .

Then, because A is to B , as B is to C , and B is equal to D ; it shall be*, as A is to B , so is D to C . * 7. 5.

But, if four right Lines be Proportionals, the Rectangle contained under the Extremes is † equal to the † 16 of this, Rectangle under the Means. Therefore the Rectan-

gle contained under A and C, is equal to the Rectangle under B and D: But the Rectangle under B and D is equal to the Square of D, for B is equal to D; wherefore the Rectangle contained under A, C, is equal to the Square of B.

And, if the Rectangle contained under A, C, be equal to the Square of B; I say, as A is to B, so is B to C.

For, the same Construction remaining, the Rectangle, contained under A and C, is equal to the Square of B; but the Square of B is the Rectangle contained under B, D; for B is equal to D; and the Rectangle contained under A, C, shall be equal to the Rectangle contained under B, D. But if the Rectangle contained under the Extremes be equal to the Rectangle contained under the Means, the four right Lines ^{† 16 of this.} shall be † Proportionals. Therefore, A is to B, as D is to C; but B is equal to D. Wherefore A is to B, as B is to C. Therefore, *if three right Lines be proportional, the Rectangle contained under the Extremes is equal to the Square of the Mean; and if the Rectangle under the Extremes be equal to the Square of the Mean, then the three right Lines are proportional; which was to be demonstrated.*

PROPOSITION XVIII.

PROBLEM.

Upon a given right Line, to describe a right-lined Figure, similar, and similarly situate, to a right-lined Figure given.

LET AB be the right Line given, and CE the right-lined Figure. It is required to describe upon the right Line AB a Figure similar, and similarly situate, to the right-lined Figure CE.

* 23. 1. Join DF, and make, * at the Points A and B, with the Line AB, the Angles GAB, ABG, each equal to the Angles C and CDF. Whence the other Angle CFD is † equal to the other Angle AGB, and so the Triangle FCD is equiangular to the Triangle GAB; and, consequently, as FD is to GB, so is † FC to GA; and so is CD to AB. Again, make ^{† 4 of this.} the Angles BGH, GBH, at the Points B and G, with

with the right Line BG, each equal to the Angles EFD, EDF; then the remaining Angle at E is †† *Cor. 32. 1.* equal to the remaining Angle at H. Therefore the Triangle FDE is equiangular to the Triangle GBH; and, consequently, as FD is to GB, so is † FE to † 4 of this. GH; and so ED to HB. But it has been proved, that FD is to GB, as FC is to GA, and as CD to AB. And therefore, as FC is to AG, so is * CD to * 11. 5. AB; and so FE to GH; and so ED to HB. And because the Angle CFD is equal to the Angle AGB; and the Angle DFE equal to the Angle BGH; the whole Angle CFE shall be equal to the whole Angle AGH. By the same Reason, the Angle CDE is equal to the Angle ABH; and the Angle at C equal to the Angle A; and the Angle E equal to the Angle H. Therefore the Figure AC is equiangular to the Figure CE; and they have the Sides about the equal Angles proportional. Consequently, the right-lined Figure AH will be † similar to the right-lined Figure † *Def. 1. of this.* CE. Therefore there is described upon the given right Line AB, the right-lined Figure AH, similar, and similarly situate, to the given right-lined Figure CE; which was to be done.

PROPOSITION XIX.

THEOREM.

Similar Triangles are in the duplicate Proportion of their homologous Sides.

LET ABC, DEF, be similar Triangles, having the Angle B equal to the Angle E; and let AB be to BC, as DE is to EF, so that BC be the Side homologous to EF. I say, the Triangle ABC, to the Triangle DEF, has a duplicate Proportion to that of the Side BC to the Side EF.

For, take * BG a third Proportion to BC and EF; * 11 of this. that is, let BC be to EF, as EF is to BG, and join GA.

Then, because AB is to BC, as DE is to EF; it shall be (by Alternation), as AB is to DE, so is BC to EF; but as BC is to EF, so is EF to BG. Therefore, as AB is to DE, so is † EF to BG; conse- † 11. 5. quently, the Sides that are about the equal Angles of

the Triangles ABG, DEF, are reciprocal: But those Triangles that have one Angle of the one equal to one Angle of the other, and the Sides about the equal Angles reciprocal, are $\dagger$ equal. Therefore the Triangle ABG is equal to the Triangle DEF; and because BC is to EF, as EF is to BG, and if three right Lines *† 15 of this.* be proportional, the first has ** Def. 10.5.* a duplicate Proportion to the third, of what it has to the second, BC to BG shall have a duplicate Proportion of that which BC has to EF; and as BC is to BG, so is the Triangle ABC to the Triangle ABG; whence the Triangle ABC bears to the Triangle ABG a duplicate Proportion to what BC doth to EF; but the Triangle ABG is equal to the Triangle DEF. Therefore the Triangle ABC, to the Triangle DEF, shall be in the duplicate Proportion of that which the Side BC has to the Side EF. Wherefore, *similar Triangles are in the duplicate Proportion of their homologous Sides*; which was to be demonstrated.

Coroll. From hence it is manifest, if three right Lines be proportional, then, as the first is to the third, so is a Triangle made upon the first to a similar and similarly described Triangle upon the second; because it has been proved, as CB is to BG, so is the Triangle ABC to the Triangle ABG, that is, to the Triangle DEF; *which was to be demonstrated.*

PROPOSITION XX.

THEOREM.

Similar Polygons are divided into similar Triangles, equal in Number, and homologous to the Wholes; and Polygon to Polygon, is in the duplicate Proportion of that which one homologous Side has to the other.

LET ABCDE, FGHLK, be similar Polygons, and let the Side AB be homologous to the Side FG. I say, the Polygons ABCDE, FGHLK, are divided into equal Numbers of similar Triangles, and homologous to the Wholes; and the Polygon ABCDE, to the Polygon FGHLK, is in the duplicate Proportion of that which the Side AB has to the Side FG.

For, let BE, EC, GL, LH, be joined.

Then,

Then, because the Polygon $ABCDE$ is similar to the Polygon $FGHKL$, the Angle BAE is equal to the Angle GFL ; and BA is to AE , as GF is to FL . Now, since ABE , FGL , are two Triangles, having one Angle of the one equal to one Angle of the other, and the Sides about the equal Angles proportional; the Triangle ABE will be * equiangular * 6 of this. to the Triangle FGL , and also similar to it. Therefore the Angle ABE is equal to the Angle FGL ; but the whole Angle ABC is † equal to the whole † 1 Def. of Angle FGH , because of the Similarity of the Poly- this. gons; therefore the remaining Angle EBC is equal to the remaining Angle LGH : And since, (by the Similarity of the Triangles ABE , FGL) as EB is to BA , so is LG to GF ; and since, also, by the Similarity of the Polygons, AB is to BC , as FG is to GH ; it shall be ‡, by Equality of Proportion, as EB is to ‡ 22. 5. BC , so is LG to GH ; that is, the Sides about the equal Angles EBC , LGH , are proportional. Wherefore the Triangle EBC is equiangular to the Triangle LGH , and, consequently, also similar to it. For the same Reason, the Triangle ECD is likewise similar to the Triangle LHK ; therefore the similar Polygons $ABCDE$, $FGHKL$, are divided into equal Numbers of similar Triangles.

I say, they are also homologous to the Wholes, that is, that the Triangles are proportional; and the Antecedents are ABE , EBC , ECD ; and their Consequents FGL , LGH , LHK . And the Polygon $ABCDE$, to the Polygon $FGHKL$, is in the duplicate Proportion of an homologous Side of the one, to an homologous Side of the other, that is, AB to FC .

For, because the Triangle ABE is similar to the Triangle FGL , the Triangle ABE shall be * to the * 19 of this. Triangle FGL , in the duplicate Proportion of BE to GL : For the same Reason, the Triangle BEC , to the Triangle GLH , is * in a duplicate Proportion of BE to GL : Therefore the Triangle ABE is † to † 11. 5. the Triangle FGL , as the Triangle BEC is to the Triangle GLH . Again, because the Triangle EBC is similar to the Triangle LGH , the Triangle EBC , to the Triangle LGH , shall be in the duplicate Proportion of the right Line CE to the right Line HL ; and so, likewise, the Triangle ECD , to the Tri-

† 12. 5.

angle LHK, shall be in the duplicate Proportion of CE to HL. Therefore the Triangle BEC is to the Triangle LGH, as the Triangle CED is to the Triangle LHK. But it has been proved, that the Triangle EBC is to the Triangle LGH, as the Triangle ABE is to the Triangle FGL: Therefore, as the Triangle ABE is to the Triangle FGL, so is the Triangle BEC to the Triangle GHL; and so is the Triangle ECD to the Triangle LHK. But as one of the Antecedents is to one of the Consequents, so are † all the Antecedents to all the Consequents. Wherefore, as the Triangle ABE is to the Triangle FGL, so is the Polygon ABCDE to the Polygon FGHLK: But the Triangle ABE, to the Triangle FGL, is in the duplicate Proportion of the homologous Side AB to the homologous Side FG; for similar Triangles are in the duplicate Proportion of the homologous Sides. Wherefore, the Polygon ABCDE, to the Polygon FGHLK, is in the duplicate Proportion of the homologous Side AB to the homologous Side FG. Therefore, *similar Polygons are divided into similar Triangles, equal in Number, and homologous to the Wholes; and Polygon to Polygon is in the duplicate Proportion of that which one homologous Side has to the other; which was to be demonstrated.*

It may be demonstrated after the same manner, that similar quadrilateral Figures are to each other in the duplicate Proportion of their homologous Sides; and this has been already proved in Triangles.



Coroll. 1. Therefore, universally, similar right-lin'd Figures are to one another in the duplicate Proportion of their homologous Sides; and if X be taken a third Proportional to AB and FG, then AB will have to X a duplicate Proportion of that which AB has to FG; and a Polygon to a Polygon, and a quadrilateral Figure to a quadrilateral Figure, will be in the duplicate Proportion of that which one homologous Side has to the other; that is, AB to FG; but this has been proved in Triangles.

2. Therefore, universally, it is manifest, if three right Lines be proportional, as the first is to the third, so is a Figure described upon the first, to a similar and similarly

similarly described Figure on the second; *which was to be demonstrated.*

PROPOSITION XXI.

THEOREM.

Figures that are similar to the same right-lined Figure, are also similar to one another.

LET each of the right-lined Figures A, B, be similar to the right-lined Figure C. I say, the right-lined Figure A is also similar to the right-lined Figure B.

For, because the right-lined Figure A is similar to the right-lined Figure C, it shall be * equiangular ^{* Def. 1. of this.} thereto; and the Sides about the equal Angles proportional. Again, because the right-lined Figure B is similar to the right-lined Figure C, it shall * be equiangular thereto; and the Sides about the equal Angles will be proportional. Therefore each of the right-lined Figures A, B, are equiangular to C, and they have the Sides about the equal Angles proportional. Wherefore the right-lined Figure A is equiangular to the right-lined Figure B; and the Sides about the equal Angles are proportional; wherefore A is similar to B; *which was to be demonstrated.*

PROPOSITION XXII.

THEOREM.

If four right Lines be proportional, the right-lined Figures, similar and similarly described upon them, shall be proportional; and if the similar right-lined Figures similarly described upon the Lines be proportional, then the right Lines shall be also proportional.

LET four right Lines AB, CD, EF, GH, be proportional; and as AB is to CD, so let EF be to GH.

Now, let the similar Figures KAB, LCD, be similarly described * upon AB, CD; and the similar * 18 of this. Figures

Figures M F, N H, similarly described upon the right Lines E F, G H. I say, as the right-lined Figure K A B is to the right-lined Figure L C D, so is the right-lined Figure M F to the right-lined Figure N H.

* 11 of this. For, take * X a third Proportional to A B, C D; and O a third Proportional to E F, G H.

* 22. 5. Then, because A B is to C D, as E F is to G H, and as C D is to X, so is G H to O; it shall be †, by Equality of Proportion, as A B is to X, so is E F to O.

† Cor. 20. of this. But A B is to X, as the right-lined Figure K A B is † to the right-lined Figure L C D; and as E F is to O, so is † the right-lined Figure M F, to the right-lined Figure N H. Therefore, as the right-lined Figure

* 11. 5. K A B is to the right-lined Figure L C D, so is * the right-lined Figure M F to the right-lined Figure N H.

And, if the right-lined Figure K A B be to the right-lined Figure L C D, as the right-lined Figure M F is to the right-lined Figure N H; I say, as A B is to C D, so is E F to G H.

† 12 of this. For, make † E F to P R, as A B is to C D, and describe upon P R a right-lined Figure S R similar, and alike situate, to either of the Figures M F and N H.

Then, because A B is to C D, as E F is to P R, and there are described upon A B, C D, similar and alike situate right-lined Figures K A B, L C D, and upon E F, P R, similar and alike situate Figures M F, S R; it shall be (by what has been already proved) as the right-lined Figure K A B is to the right-lined Figure L C D, so is the right-lined Figure M F to the right-lined Figure S R: But (by the *Hyp.*) as the right-lined Figure K A B is to the right-lined Figure L C D, so is the right-lined Figure M F to the right-lined Figure N H. Therefore, as the right-lined Figure M F is to the right-lined Figure N H, so is the right-lined Figure M F to the right-lined Figure S R: And since the right-lined Figure M F has the same Proportion to N H, as it hath to S R, the right-lined Figure N H shall be † equal to the right-lined Figure S R; it is also similar to it, and alike described; therefore G H is equal to P R. And because A B is to C D, as E F is to P R; and P R is equal to G H; it shall be, as A B is to C D, so is E F to G H. Therefore, if four right Lines be proportional, the right-lined Figures, similar and similarly described upon them, shall be proportional; and

† 9. 5.

and if the similar right-lined Figures, similarly described upon the Lines, be proportional, then the right Lines shall also be proportional; which was to be demonstrated.

L E M M A.

Any three right Lines A, B, and C, being given, the Ratio of the first A, to the third C, is equal to the Ratio compounded of the Ratio of the first A to the second B, and of the Ratio of the second B to the third C.

FOR Example, let the Number 3 be the Exponent or Denominator of the Ratio of A to B; that is, let A be three times B, and let the Number 4 be the Exponent of the Ratio of B to C; then the Number 12, produced by the Multiplication of 4 and 3, is the compounded Exponent of the Ratio of A to C: For, since A contains B thrice, and B contains C four times, A will contain C thrice four times, that is, 12 times. This is also true of other Multiples, or Submultiples; but this Theorem may be universally demonstrated thus: The Quantity of the Ratio of A to

B, is the Number $\frac{A}{B}$ viz. which, multiplying the Consequent, produces the Antecedent. So likewise the Quantity of the Ratio of B to C, is $\frac{B}{C}$. And these two Quantities, multiplied by each other, produce the Number $\frac{A \times B}{B \times C}$, which is the Quantity of the Ratio,

that the Rectangle comprehended under the right Lines A and B, has to the Rectangle comprehended under the right Lines B and C; and so the said Ratio of the Rectangle under A and B, to the Rectangle under B and C, is that which, in the Sense of Def. 5. of this Book, is compounded of the Ratios of A to B, and B to C; but (by 1. 6.) the Rectangle contained under A and B, is to the Rectangle contained under B and C, as A is to C; therefore the Ratio of A to C is equal to the Ratio compounded of the Ratios of A to B, and of B to C.



If

If any four right Lines A, B, C, and D, be proposed, the Ratio of the first A to the fourth D, is equal to the Ratio compounded of the Ratio of the first A to the second B, and of the Ratio of the second B to the third C, and of the Ratio of the third C to the fourth D.

For, in three right Lines A, C, and D, the Ratio of A to D is equal to the Ratios compounded of the Ratios of A to C, and of C to D; and it has been already demonstrated, that the Ratio of A to C is equal to the Ratio compounded of the Ratios of A to B, and B to C. Therefore the Ratio of A to D is equal to the Ratio compounded of the Ratios of A to B, of B to C, and of C to D. After the same manner we demonstrate, in any Number of right Lines, that the Ratio of the first to the last is equal to the Ratio compounded of the Ratios of the first to the second, of the second to the third, of the third to the fourth, and so on to the last.

This is true of any other Quantities besides right Lines, which will be manifest, if the same Number of right Lines A, B, C, &c. as there are Magnitudes, be assumed in the same Ratio, viz. so that the right Line A is to the right Line B, as the first Magnitude is to the second, and the right Line B to the right Line C, as the second Magnitude is to the third, and so on. It is manifest (by 22. 5.) by Equality of Proportion, that the first right Line A is to the last right Line, as the first Magnitude is to the last; but the Ratio of the right Line A to the last right Line is equal to the Ratio compounded of the Ratios of A to B, B to C, and so on to the last right Line: But (by the Hyp.) the Ratio of any one of the right Lines to that nearest to it, is the same as the Ratio of a Magnitude of the same Order to that nearest it. And therefore the Ratio of the first Magnitude to the last is equal to the Ratio compounded of the Ratios of the first Magnitude to the second, of the second to the third, and so on to the last; which was to be demonstrated.

PROPOSITION XXIII.

THEOREM.

Equiangular Parallelograms have the Proportion to one another that is compounded of their Sides.

LET AC, CF, be equiangular Parallelograms, having the Angle BCD equal to the Angle ECG. I say, the Parallelogram AC, to the Parallelogram CF, is in the Proportion compounded of their Sides, viz. compounded of the Proportion of BC to CG, and of DC to CE.

For, let BC be placed in the same right Line with CG.

Then DC shall be * in a strait Line with CE, * 14. 1. and complete the Parallelogram DG; and then †, as † 12 of this, BC is to CG, so is some right Line K to L; and as DC is to CE, so let L be to M.

Then the Proportions of K to L, and of L to M, are the same as the Proportions of the Sides, viz. of BC to CG, and DC to CE; but the Proportion of K to M is † compounded of the Proportion of K to L, and of the Proportion of L to M. Wherefore, ^{† Lemma preced.} also, K to M hath a Proportion compounded of the Sides. Then, because BC is to CG as the Parallelogram AC is * to the Parallelogram CH: And since * 1 of this, BC is to CG, as K is to L, it shall be †, as K is to L, † 11. 5. * so is the Parallelogram AC to the Parallelogram CH. Again, because DC is to CE, as the Parallelogram CH is to the Parallelogram CF; and since as DC is to CE, so is L to M; therefore, as L is to M, so shall † the Parallelogram CH be to the Parallelogram CF; and, consequently, since it has been prov'd that K is to L, as the Parallelogram AC is to the Parallelogram CH, and as L is to M, so is the Parallelogram CH to the Parallelogram CF; it shall be †, by Equality of Proportion, as K is to M, so is † 22. 5. the Parallelogram AC to the Parallelogram CF; but K to M hath a Proportion compounded of the Sides: Therefore, also, the Parallelogram AC, to the Parallelogram CF, hath a Proportion compounded of the Sides. Wherefore, *equiangular Parallelograms have the*

the Proportion to one another that is compounded of their Sides ; which was to be demonstrated.

PROPOSITION XXIV.

THEOREM.

In every Parallelogram, the Parallelograms that are about the Diameter, are similar to the Whole, and also to one another.

LET ABCD be a Parallelogram, whose Diameter is AC; and EG, HK, be Parallelograms about the Diameter AC. I say, the Parallelograms EG, HK, are similar to the Whole ABCD, and also to each other.

For, because EF is drawn parallel to BC, the Side of the Triangle ABC, it shall be*, as BE to EA, so is CF to FA. Again, because FG is drawn parallel to CD, the Side of the Triangle ACD, it shall be, as CF to FA, so is* DG to GA. But CF is to FA, (as has been proved) as BE is to EA. Therefore, as BE is to EA, so is† DG to GA; and by compounding, as BA is to AE, so is† DA to AG; and, by Alternation, as BA is to AD, so is EA to AG. Therefore the Sides of the Parallelograms ABCD, EG, which are about the common Angle BAD, are proportional. And because GF is parallel to DC, the Angle AGF is* equal to the Angle ADC, and the Angle GFA equal to the Angle DCA; and the Angle DAC is common to the two Triangles ADC, AGF. Wherefore the Triangle ADC will be equiangular to the Triangle AGF. For the same Reason, the Triangle ACB is equiangular to the Triangle AFE. Therefore the whole Parallelogram ABCD is equiangular to the Parallelogram EG; and so AD is to DC, as AG is† to GF. But DC is to CA, as GF is to FA; and AC is to CB, as AF is to FE; and, moreover, CB is to BA, as FE is to EA. Wherefore, since it has been proved, that DC is to CA, as GF is to FA; and AC is to CB, as AF is to FE; it shall be, by Equality of Proportion, as DC is to CB, so is GF to FE. Therefore the Sides that are about the equal Angles of the Parallelograms ABCD,

* 2 of this.

† 11. 5.

† 18. 5.

* 29. 9.

† 4 of this.

ABCD, EG, are proportional; and, accordingly, the Parallelogram ABCD is similar to the Parallelogram EG. For the same Reason, the Parallelogram ABCD is similar to the Parallelogram KH. Therefore both the Parallelograms EG, HK, are similar to the Parallelogram ABCD. But right-lined Figures that are similar to the same * right-lined Figure, are similar to one another. Therefore the Parallelogram EG is similar to the Parallelogram HK. And so, *in every Parallelogram, the Parallelograms that are about the Diameter are similar to the Whole, and also to one another; which was to be demonstrated.*

PROPOSITION XXV.

PROBLEM.

To describe a right-lined Figure similar to a right-lined Figure which shall be given, and equal to another right-lined Figure given.

LET ABC be a given right-lined Figure, to which it is required to describe another similar and equal to D.

On the Side BC of the given Figure ABC, * make * 44. 1. the Parallelogram BE equal to the right-lined Figure ABC; and on the Side CE make * the Parallelogram CM equal to the right-lined Figure D, in the Angle FCE, equal to the Angle CBL. Then BC, CF, as also LE, EM, will be † in two strait Lines. † 14. 1. Find † GH a mean Proportional between BC, CF; † 13 of this. and on GH let there be described * the right-lined * 18 of this. Figure KGH, similar, and alike situate, to the right-lined Figure ABC.

And then, because BC is to GH, as GH is to CF; and since, when three right Lines are proportional, the first is to the third as the Figure described on the first is † to a similar and alike situate Figure described on † Cor. 20. the second, it shall be, as BC is to CF, so is the right-lined Figure ABC to the right-lined Figure KGH. But as BC is to CF, so is † the Parallelogram BE to † 1 of this. the Parallelogram EF. Therefore, as the right-lined Figure ABC is to the right-lined Figure KGH, so is the Parallelogram BE to the Parallelogram EF.

Where-

Wherefore, (by Alternation) as the right-lined Figure ABC is to the Parallelogram BE, so is the right-lined Figure KGH to the Parallelogram EF. But the right-lined Figure ABC is equal to the Parallelogram BE. Therefore the right-lined Figure KGH is also equal to the Parallelogram EF. But the Parallelogram EF is equal to the right-lined Figure D. Therefore the right-lined Figure KGH is equal to D. But KGH is similar to ABC. Consequently, there is described the right-lined Figure KGH similar to the given Figure ABC, and equal to the given Figure D; which was to be done.

PROPOSITION XXVI.

THEOREM.

If from a Parallelogram be taken away another similar to the Whole, and in like manner situate, having also an Angle common with it, then is that Parallelogram about the same Diameter with the Whole.

LET the Parallelogram AF be taken away from the Parallelogram ABCD, similar to ABCD, and in like manner situate, having the Angle DAB common. I say, the Parallelogram ABCD is about the same Diameter with the Parallelogram AF.

For, if it be not, let AHC be the Diameter of the Parallelogram BD, and let GF be produced to H; also, let HK be drawn parallel to AD, or BC.

Then, because the Parallelogram ABCD is about the same Diameter as the Parallelogram KG, the Parallelogram ABCD shall be * similar to the Parallelogram KG; and so, as DA is to AB, so is † GA to AK. But because of the Similarity of the Parallelograms ABCD, EG, as DA is to AB, so is GA to AE. And therefore, as GA is † to AE, so is GA to AK. And since GA has the same Proportion to AK as to AE, AE is † equal to AK, the less to a greater, which is absurd. Therefore the Parallelogram ABCD is not about the same Diameter as the Parallelogram AH. And therefore it will be about the same Diameter with the Parallelogram AF. Therefore,

if from a Parallelogram be taken away another similar

* 2 of this.

* 1 Def. of this.

† 11. 5.

‡ 9. 5.

to the *Whole*, and in like manner situate, having also an *Angle* common with it, then is that *Parallelogram* about the same *Diameter* with the *Whole*; which was to be demonstrated.

PROPOSITION XXVII.

THEOREM.

Of all Parallelograms applied to the same right Line, and wanting in Figure by Parallelograms similar and alike situate, described on the half Line, the greatest is that which is applied to the half Line, being similar to the Defect.

LET *AB* be a right Line, bisected in the Point *C*; and let the *Parallelogram AD* be applied to the right Line *AB*, wanting in Figure the *Parallelogram CE*, similar and alike situate to that describ'd on half of the right Line *AB*. I say, *AD* is the greatest of all *Parallelograms* applied to the right Line *AB*, wanting in Figure by *Parallelograms* similar and alike situate to *CE*. For, let the *Parallelogram AF* be applied to the right Line *AB*, wanting in Figure the *Parallelogram HK*, similar and alike situate to the *Parallelogram CE*. I say, the *Parallelogram AD* is greater than the *Parallelogram AF*.

For, because the *Parallelogram CE* is similar to the *Parallelogram HK*, they stand * about the same Dia- * 26 of this. meter. Let *DB*, their *Diameter*, be drawn, and the Figure described; then, since the *Parallelogram CF* is † † 43. 1. equal to *FE*, let *HK*, which is common, be added; and the Whole *CH* is equal to the Whole *KE*. But *CH* is † equal to *CG*, because the right Line *AC* is † 36. 1. equal to *CB*. Therefore the Whole *AF* is equal to the *Gnomon LNM*; and so *CE*, that is, the *Parallelogram AD*, is greater than the *Parallelogram AF*. Therefore, *of all Parallelograms applied to the same right Line, and wanting in Figure by Parallelograms similar and alike situate, described on the half Line, the greatest is that which is applied to the half Line, being similar to the Defect*; which was to be demonstrated.

PROPOSITION XXVIII.

PROBLEM.

To a right Line given to apply a Parallelogram equal to a right-lined Figure given, deficient by a Parallelogram, which is similar to another given Parallelogram; but it is necessary, that the right-lined Figure given, to which the Parallelogram to be applied must be equal, be not greater than the Parallelogram which is applied to the half Line, since the Defects must be similar, viz. the Defect of the Parallelogram applied to the half Line, and the Defect of the Parallelogram to be applied.

LET AB be a given right Line, and let the given right-lined Figure, to which the Parallelogram to be applied to the right Line AB must be equal, be C, which must not be greater than the Parallelogram applied to the half Line, the Defects being similar; and let D be the Parallelogram, to which the Defect of the Parallelogram to be applied is similar. Now it is required to apply a Parallelogram equal to the given right-lined Figure C to the given right Line AB, deficient by a Parallelogram similar to D.

* 18 of this. Let AB be bisected in E, and on EB describe * the Parallelogram E B F G, similar and alike situate to D, and complete the Parallelogram A G.

Now, A G is either equal to C, or greater than it, because of the Determination. If A G be equal to C, what was proposed will be done; for the Parallelogram A G is applied to the right Line AB, equal to the given right-lined Figure C, deficient by the Parallelogram E F, similar to the Parallelogram D. But, if it be not equal, then H E is greater than C; but E F is equal to H E, therefore E F shall also be greater than C. Now make † the Parallelogram K L M N similar and alike situate to D, and equal to the Excess, by which E F exceeds C. But D is similar to E F; wherefore K M shall also be similar to E F. Therefore let the right Line K L be homologous to G E, and L M to G F: Then, because E F is equal to C and K M together, E F will be greater than K M; and

† 25 of this.

and so the right Line GE is greater than KL, and GF than LM. Make GX equal to KL, and GO equal to LM, and complete the Parallelogram XGOP. Therefore XO is equal and similar to KM; but KM is similar to EF; therefore XO is * ^{21 of this.} similar to EF; and so XO is † about the same Diameter with FE: Let GPB be their Diameter, and the Figure be described.

Then, since EF is equal to C and KM together, and XO is equal to KM, the Gnomon $\Upsilon\Phi\Psi$ remaining is equal to the remaining Figure C; and because OR is equal to XS, let SR, which is common, be added; then the Whole OB is equal to the Whole XB; but XB is equal to TE, since the Side AE is equal to the Side EB. Wherefore TE is equal to OB. Add XS, which is common, and then the whole TS is equal to the whole Gnomon $\Upsilon\Phi\Psi$; but the Gnomon $\Upsilon\Phi\Psi$ has been proved equal to C; and TS shall be equal to C; and so the Parallelogram TS is applied to the right Line AB, equal to the given right-lined Figure C, and deficient by a Parallelogram SR, similar to the Parallelogram D, because SR is similar to FE; *which was to be done.*

PROPOSITION XXIX.

THEOREM.

To a right Line given, to apply a Parallelogram equal to a right-lined Figure given, exceeding by a Parallelogram, which shall be similar to another given Parallelogram.

LET AB be a given right Line, and let C be the given right-lined Figure, to which that to be applied to AB must be equal: Likewise, let D be the Parallelogram, to which the exceeding Parallelogram is to be similar; it is required to apply a Parallelogram to the right Line AB, equal to the given right-lined Figure C, exceeding by a Parallelogram similar to D.

Bisect AB in E, and let the Parallelogram EL be described * upon the right Line EB, similar and alike * ^{18. 1.} situate to D; and that † the Parallelogram GH, equal † ^{25 of this.}

to EL and C together, but similar to D , and alike situate; therefore GH is similar to EL . Let KH be a Side homologous to FL , and KG to FE ; then, because the Parallelogram GH is greater than the Parallelogram EL , the right Line KH will be greater than FL , and KG greater than FE . Let FL , FE , be produc'd, and let FLM be equal to KH , FEN , equal to KG , and complete the Parallelogram MN ; therefore MN is equal and similar to GH ; but GH is similar to EL , and so MN shall be $\dagger$ similar to EL ; and, accordingly, EL is $*$ about the same Diameter with MN . Let FX be their Diameter, and describe the Figure.

$\dagger$ 1 of this.

$*$ 26 of this.

Then, since GH is equal to EL and C together, as, likewise, to MN ; therefore MN shall be equal to EL and C . Let EL , which is common, be taken away; then the Gnomon $\Upsilon\Phi\Psi$ remaining is equal to C ; and since AE is equal to EB , the Parallelogram AN will be also equal to the Parallelogram EP , that is, to LO ; and if EX , which is common, be added, then the whole Parallelogram AX is equal to the Gnomon $\Upsilon\Phi\Psi$; but the Gnomon $\Upsilon\Phi\Psi$ is equal to C ; therefore AX shall be also equal to C . Wherefore the Parallelogram AX is applied to the given right Line AB , equal to the given right-lin'd Figure C , and exceeding by the Parallelogram PO , similar to the Parallelogram D ; which was to be done.

PROPOSITION XXX.

PROBLEM.

To cut a given terminate right Line according to extreme and mean Ratio.

LET AB be a given terminate Line; it is requir'd to cut the same according to extreme and mean Ratio.

$*$ 46. 1.

$\dagger$ 29 of this.

Describe $*$ BC , the Square of AB ; and apply the Parallelogram CD to AC , equal to the Square BC , exceeding $\dagger$ by the Figure AD similar to BC ; but BC is a Square, therefore AD shall also be a Square.

Now, because BC is equal to CD , take away CE , which is common; then BF remaining shall be equal to

to AD remaining; but BF is equiangular to AD; therefore the Sides that are about the equal Angles are † reciprocally proportional; and so, as FE is to † ^{14 of this.} ED, so is AE to EB; but FE is * equal to AC, that * ^{34. 1.} is, to AB; and ED to AE: Wherefore, as BA is to AE, so is AE to EB; but AB is greater than AE; therefore AE is † greater than EB; and so the right † ^{14. 5.} Line AB is cut according to the extreme and mean Ratio in the Point E; and AE is the greater Segment thereof; *which was to be done.*

Otherwise thus: Let AB be the right Line given; it is required to cut the same into extreme and mean Ratio.

Divide † AB so in C, that the Rectangle contained † ^{11. 2.} under AB, BC, be equal to the Square of AC.

Then, because the Rectangle under AB, BC, is equal to the Square of AC, it shall be *, as BA is * ^{17 of this.} to AC, so is AC to CB; and so the right Line AB is cut into mean and extreme Ratio; *which was to be done.*

PROPOSITION XXXI.

THEOREM.

Any Figure described upon the Side of a right-angled Triangle subtending the right Angle, is equal to the Figures described upon the Sides containing the right Angle, being similar and alike situate to the former Figure.

LET ABC be a rectangular Triangle, having the right Angle BAC. I say, the Figure described on BC, is equal to the two Figures together described on BA, AC, which are similar and alike situate to the Figure described on BC.

For, draw the Perpendicular AD.

Then, because the right Line AD is drawn in the right-angled Triangle ACB, from the right Angle A, perpendicular to the Base BC; the Triangles ABD, ADC, which are about the Perpendicular AD, will be * similar to the whole Triangle ABC, and also to * ^{8 of this.} each other. Then, because the Triangle ABC is similar to the Triangle ABD, it shall be *, as CB is to BA,

† Cor. 20.
of this.

24. 5.

so is BA to BD: And since, when three right Lines are proportional, the first shall be † to the third, as a Figure described on the first, to a similar and alike situate Figure described on the second; therefore, as C is to BD, so is a Figure described on CB to a similar and alike situate Figure described on BA. For the same Reason, as BC is to CD, so is a Figure described on BC to one described on CA. Wherefore, also, as BC is to BD and DC together, so is † the Figure described on BC, to those two together, that are described similar and alike situate on BA, AC; but BC is equal to BD and DC together: Therefore the Figure described on BC is equal to those together described on BA, AC, similar and alike situate to that on BC. Wherefore, any Figure described upon the Side of a right-angled Triangle subtending the right Angle, is equal to the Figures described upon the Sides containing the right Angle, being similar and alike situate to the former Figure; which was to be demonstrated.

PROPOSITION XXXII.

THEOREM.

If two Triangles, having two Sides proportional to two Sides, be so compounded, or set together, at one Angle, that their homologous Sides be parallel, then the other Sides of these Triangles will be in one strait Line,

LET there be two Triangles ABC, DCE, having two Sides BA, AC, of the one, proportional to two Sides CD, DE, of the other; viz. let BA be to AC, as CD is to DE; also, let AB be parallel to DC, and AC to DE. I say, BC, CE, are both in one strait Line.

29. 1.

For, because AB is parallel to DC, and the right Line AC falls on them, the alternate Angles BAC, ACD, will be * equal to each other. And by the same Reason, the Angle CDE is equal to the Angle ACD; wherefore the Angle BAC is equal to the Angle CDE. Then, because ABC, DCE, are two Triangles, having one Angle A equal to one Angle D,

D, and the Sides about the equal Angles proportional, viz. BA to AC, as CE to DE; the Triangle ABC will be * equiangular to the Triangle DCE; where- * 6 of this, fore the Angle ABC is equal to the Angle DCE; but the Angle ACD has been proved to be equal to the Angle BAC; therefore the whole Angle ACE is equal to the two Angles ABC, BAC; and if ACB, which is common, be added, then the Angles ACE, ACB, are equal to the Angles BAC, ACB, CBA; but the Angles BAC, ACB, CBA, are equal to two right Angles. Therefore the Angles ACE, ACB, will also be equal to two right Angles; and so at the Point C, in the right Line AC, two right Lines BC, CE, tending contrary ways, make the adjacent Angles ACE, ACB, equal to two right Angles; therefore BC shall be † in the same right Line with CE. † 14. 1. Wherefore, if two Triangles, having two Sides proportional to two Sides, be so compounded, or set together, at one Angle, that their homologous Sides be parallel, then the other Sides of these Triangles will be in one strait Line; which was to be demonstrated.

PROPOSITION XXXIII.

THEOREM.

In equal Circles the Angles have the same Proportion with their Circumferences on which they stand, whether the Angles be at the Centres, or at the Circumferences; and so likewise are the Sectors, as being at the Centres.

LET ABC, DEF, be equal Circles; and let the Angles BGC, EHF, be at their Centres G, H; and the Angles BAC, EDF, at their Circumferences. I say, as the Circumference BC is to the Circumference EF, so is the Angle BGC to the Angle EHF; and so is the Angle BAC to the Angle EDF; and so is the Sector BGC to the Sector EHF.

For, assume any Number of continuous Circumferences CK, KL, each equal to BC; and also any Number FM, MN, each equal to EF; and join GK, GL, HM, HN.

N 4

Then,

Then, because the Circumferences BC , CK , KL , are equal to each other, the Angles BGC , CGK , KGL , will be * also equal to one another; and so the Circumference BL is the same Multiple of the Circumference BC , as the Angle BGL is of the Angle BGC . For the same Reason, the Circumference NE is the same Multiple of the Circumference EF , as the Angle ENH is of the Angle EHF ; but if the Circumference BL be equal to the Circumference EN , then the Angle BGL shall be equal to the Angle ENH ; and if the Circumference BL be greater than the Circumference EN , the Angle BGL will be greater than the Angle ENH ; and, if less, less. Therefore here are four Magnitudes, viz. the two Circumferences BC , EF , and the two Angles BGC , EHF ; and since there are taken Equimultiples of the Circumference BC , and the Angle BGC , to wit, the Circumference BL , and the Angle BGL ; as, also, Equimultiples of the Circumference EF , and the Angle EHF , viz. the Circumference EN , and the Angle ENH ; and because it is proved, if the Circumference BL exceeds the Circumference EN , the Angle BGL will likewise exceed the Angle ENH ; and, if equal, equal; if less, less; it shall be, as the Circumference BC is to the Circumference EF , so is † the Angle BGC to the Angle EHF ; but as the Angle BGC is to the Angle EHF , so is ‡ the Angle BAC to the Angle EDF ; for the former are * double to the latter: Therefore, as the Circumference BC is to the Circumference EF , so is the Angle BGC to the Angle EHF ; and so the Angle BAC to the Angle EDF .

† Def. 5. 5.

‡ 15. 5.

* 20. 3.

Wherefore, in equal Circles, Angles have the same Proportion as the Circumferences they stand on, whether they be at the Centres, or at the Circumferences.

I say, moreover, that as the Circumference BC is to the Circumference EF , so is the Sector GBC to the Sector HFE .

For, join BC , CK , and assume the Points X , O , in the Circumferences BC , CK ; and join BX , XC , CO , OK .

‡ 4. 1.

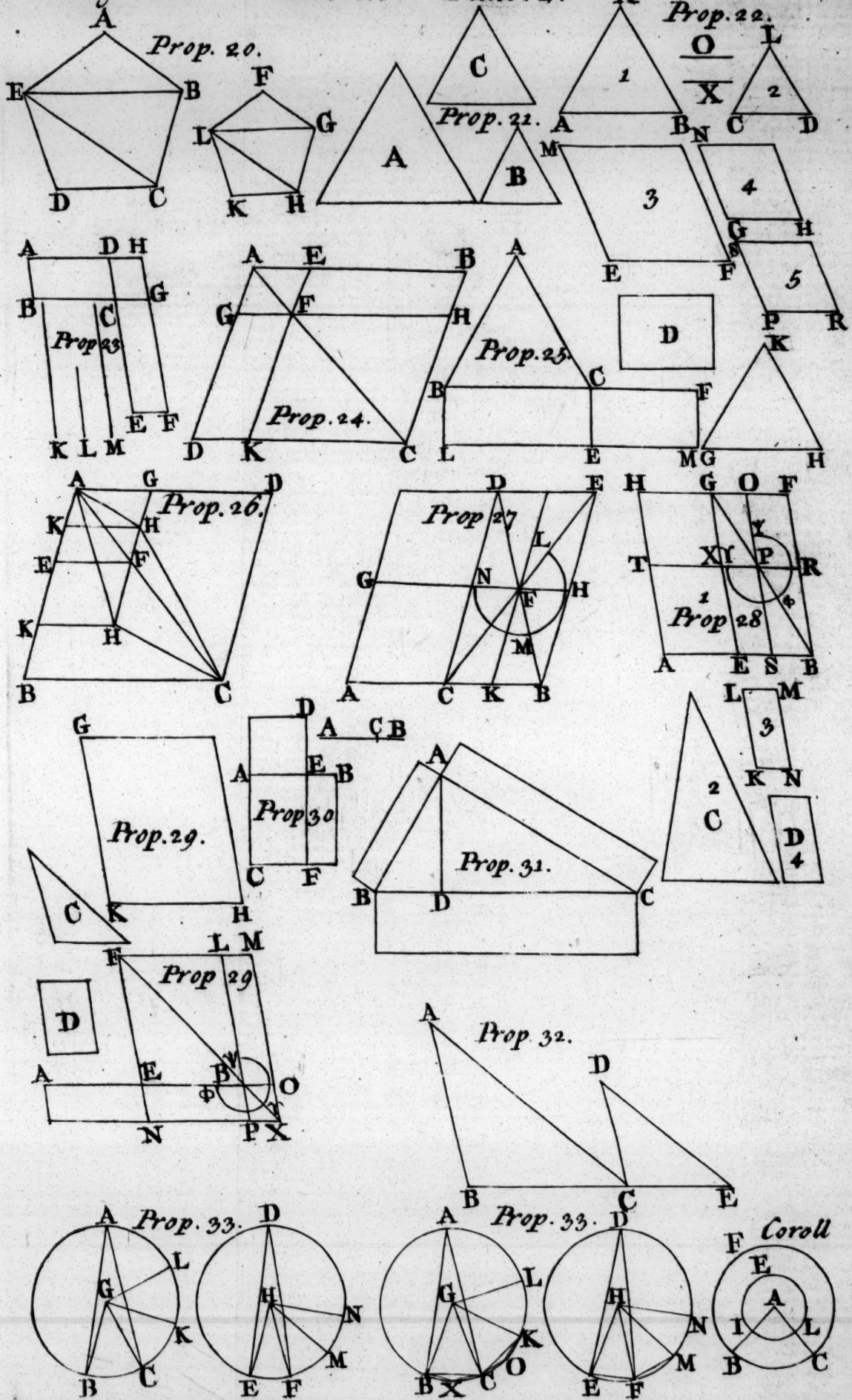
Then, because the two Sides BG , GC , are equal to the two Sides CG , GK , and they contain equal Angles, the Base BC shall be † equal to the Base CK .

CK; as likewise the Triangle GBC to the Triangle GCK. And, because the Circumference BC is equal to the Circumference CK, and the Circumference remaining, which makes up the whole Circle ABC, is equal to the remaining Circumference, which makes up the same Circle, the Angle BXC is equal to the Angle COX; and so the Segment BXC is similar to the Segment COK; and they are upon equal right Lines BC, CK; but similar Segments of Circles that stand upon equal right Lines, are * equal to each * 24. 3. other: Therefore the Segment BXC is equal to the Segment COK. But the Triangle BGC is also equal to the Triangle CGK, and so the whole Sector BGC will be equal to the whole Sector CGK. By the same Reason the Sector GKL will be equal to the Sector GBC, or GCK; therefore the three Sectors BGC, CGK, KGL, are equal to one another; so, likewise, are the Sectors HEF, HFM, HMN. Wherefore the Circumference LB is the same Multiple of the Circumference BC, as the Sector GBL is of the Sector GBC. For the same Reason, the Circumference NE is the same Multiple of the Circumference EF, as the Sector HEN is of the Sector HEF; but if the Circumference BL be equal to the Circumference EN, then the Sector BGL will be equal to the Sector EHN; and if the Circumference BL exceeds the Circumference EN, then the Sector BGL will also exceed the Sector EHN; and, if less, less. Therefore, since there are four Magnitudes, to wit, the two Circumferences BC, EF, and the two Sectors GBC, EHF; and there are taken of the Circumference BL, and the Sector GBL, Equimultiples of the Circumference CB, and the Sector CGB; as also of the Circumference EN, and the Sector HEN, Equimultiples of the Circumference EF, and the Sector HEF; and because it is proved, that, if the Circumference BL exceeds the Circumference EN, the Sector BGL will also exceed the Sector EHN; and, if equal, equal; if less, less; therefore, as the Circumference BC is to the Circumference EF, so is the Sector GBC to the Sector HEF; *which was to be demonstrated.*

Coroll. 1. An Angle at the Centre of a Circle is to four right Angles, as an Arc on which it stands is to the whole Circumference; for as the Angle BAC is to a right Angle, so is the Arc BC to a Quadrant of the Circle: Wherefore, if the Consequents be quadrupled, the Angle BAC shall be to four right Angles, as the Arc BC is to the whole Circumference.

2. The Arcs IL , BC , of unequal Circles, which subtend equal Angles, whether at their Centres, or Circumferences, are similar; for IL is to the whole Circumference ILE , as the Angle IAL is to four right Angles; but as IAL , or BAC , is to four right Angles, so is the Arc BC to the whole Circumference BCF . Therefore, as IL is to the whole Circumference ILE , so is BC to the whole Circumference BCF ; and so the Arcs IL , BC , are similar.
3. Two Semidiameters AB , AC , cut off similar Arcs IL , BC , from concentric Circumferences.

The END of the SIXTH BOOK,





E U C L I D's ELEMENTS.

B O O K XI.

DEFINITIONS.

- I. **A** Solid is that which has Length, Breadth, and Thickness.
- II. The Term of a Solid is a Superficies.
- III. A Right Line is perpendicular to a Plane, when it makes Right Angles with all the Lines that touch it, and are drawn in the said Plane.
- IV. A Plane is perpendicular to a Plane, when the Right Lines in one Plane, drawn at Right Angles to the common Section of the two Planes, are at Right Angles to the other Plane.
- V. The Inclination of a Right Line to a Plane, is the acute Angle contained under that Line, and another Right one drawn in the Plane from that End of the inclining Line which is in the Plane, to the Point where a Right Line falls from the other End of the inclining Line perpendicular to the Plane.
- VI. The Inclination of a Plane to a Plane, is the acute Angle contained under the Right Lines drawn

drawn in both the Planes to the same Point at their common Intersection, and making Right Angles with it.

VII. Planes are said to be inclined similarly when the said Angles of Inclination are equal.

VIII. Parallel Planes are such, which, being produced, never meet.

IX. Similar solid Figures are such as are contained under equal Numbers of similar Planes.

X. Equal and similar solid Figures are those that are contained under equal Numbers of similar and equal Planes.

XI. A solid Angle is the Inclination of more than two Right Lines that touch one another, and are not in the same Superficies: Or, a solid Angle is that which is contained under more than two plane Angles, which are not in the same Superficies, but being all at one Point.

XII. A Pyramid is a solid Figure comprehended under divers Planes set upon one Plane, and put together at one Point.

XIII. A Prism is a solid Figure contained under three Planes, whereof the two opposite are equal, similar, and parallel, and the others Parallelograms.

XIV. A Sphere is a solid Figure, made when a Diameter of a Semicircle remaining at Rest, the Semicircle is turned about till it returns to the same Place from whence it began to move.

XV. The Axis of a Sphere is that fixed Line about which the Semicircle is turned.

XVI. The Centre of a Sphere is the same with that of the Semicircle.

XVII. The Diameter of a Sphere is a Right Line drawn through the Centre, and terminated on either Side by the Superficies of the Sphere.

XVIII. A Cone is a Figure described, when one of the Sides of a Right-angled Triangle, containing

the Right Angle, remaining fixed, the Triangle is turned about till it returns to the Place from whence it first began to move. And if the fixed Right Line be equal to the other that contains the Right Angle, then the Cone is a rectangular Cone; but, if it be less, it is an obtuse-angled Cone; if greater, an acute-angled Cone.

XIX. The Axis of a Cone is that fixed Right Line, about which the Triangle is moved.

XX. The Base of a Cone is the Circle described by the Right Line moved about.

XXI. A Cylinder is a Figure described by the Motion of a Right-angled Parallelogram, one of the Sides containing the Right Angle, remaining fixed while the Parallelogram is turned about to the same Place from whence it began to be moved.

XXII. The Axis of a Cylinder is that fixed Right Line, about which the Parallelogram is turned.

XXIII. And the Bases of a Cylinder are the Circles that are described by the Motion of the two opposite Sides of the Parallelogram.

XXIV. Similar Cones and Cylinders are such, whose Axes and Diameters of their Bases are proportional.

XXV. A Cube is a solid Figure contained under six equal Squares.

XXVI. A Tetrabedron is a solid Figure contained under four equal equilateral Triangles.

XXVII. An Octabedron is a solid Figure contained under eight equal equilateral Triangles.

XXVIII. A Dodecabedron is a solid Figure contained under twelve, equal equilateral and equiangular Pentagons.

XXIX. An Icosabedron is a solid Figure contained under twenty equal equilateral Triangles.

XXX. A Parallelopipedon is a Figure contained under six quadrilateral Figures, whereof those which are opposite are parallel.

PROPOSITION I.

THEOREM.

One Part of a right Line cannot be in a plane Superficies, and another Part above it.

FOR, if possible, let the Part AB of the right Line ABC , be in a plane Superficies, and the Part BC above the same.

There will be some right Line in the aforesaid Plane, which, with AB , will be but one strait Line. Let this Line be DB .

Then the two given right Lines ABC , ABD have one common Segment AB , which is impossible, for one right Line will not meet another in more than one Point. Wherefore, *one Part of a right Line cannot be in a plane Superficies, and another Part above it*; which was to be demonstrated.

PROPOSITION II.

THEOREM.

If two right Lines cut each other, they are both in one Plane, and every Triangle is in one Plane.

LET two right Lines AB , CD , cut each other in the Point E . I say, they are both in one Plane, and every Triangle is in one Plane.

For, take any Points, F and G , in the right Lines AB , CD ; and join CB , FG ; and let there be drawn FH , GK . In the first Place, I say, the Triangle EBC is in one Plane.

For, if one Part FHC , or GBK , of the Triangle EBC , be in one Plane, and the other Part in another Plane; then one Part of each of the Lines EB , EC , shall be in one Plane, and the other Part in another Plane, which we have proved * to be absurd. Therefore the Triangle EBC is one Plane; but both right Lines EC , EB , are in the same Plane as the Triangle EBC .

* 1 of this.

Triangle BCE is; and AB, CD, are both in the same Plane as EC, EB, are. Wherefore, the right Lines AB, CD, are both in one Plane, and every Triangle is in one Plane; *which was to be demonstrated.*

PROPOSITION III.

THEOREM.

If two Planes cut each other, their common Section will be a right Line.

LET two Planes AB, CB, cut each other, whose common Section is the Line DB. I say, DB is a right Line.

For, if it be not, draw the right Line DEB in the Plane AB, from the Point D to the Point B, and the right Line DFB in the Plane BC.

Then two right Lines DEB, DFB, have the same Terms, and include a Space, which is * absurd. * *AXIOM 10.* Therefore DEB, DFB, are not right Lines. In the same manner we demonstrate, that no other Line, drawn from the Point D to the Point B, is a right Line, besides DB, the common Section of the Planes AB, BC. *If, therefore, two Planes cut each other, their common Section will be a right Line; which was to be demonstrated.*

PROPOSITION IV.

THEOREM.

If to two right Lines, cutting one another, a third stands at right Angles in the common Section, it shall be also at right Angles to the Plane drawn through the said Lines.

LET the right Line EF stand at right Angles to the two right Lines AB, CD, in the common Section E. I say, EF is also at right Angles to the Plane drawn through AB, CD.

For,

For, take the right Lines EA, EB, CE, DE equal; and thro' E any-how draw the right Line GEH, and join AD, CB; and from the Point E let there be drawn FA, FG, FD, FC, FH, FE. Then, because two right Lines AE, ED, are equal to two right Lines CE, EB, and they contain * the equal Angles AED, CEB, the Base AD shall be equal to the Base CB, and the Triangle AED equal to the Triangle CEB; and so, likewise, is the Angle DAE equal to the Angle ECB; but the Angle AEG is * equal to the Angle BEH; therefore AGH, BEH, are two Triangles, having two Angles of the one equal to the two Angles of the other, each to each, and one Side AE equal to one Side EB, viz. those that are at the equal Angles; and so the other Sides of the one will be † equal to the other Sides of the other. Therefore GE is equal to EH, and AG equal to BH; and since AE is equal to EB, and FE is common, and at right Angles, the Base AF shall be equal to the Base FB: For the same Reason, likewise shall CF be equal to FD. Again, because AD is equal to CB, and AF to FB, the two Sides FA, AD, will be equal to the two Sides FB, BC, each to each; but the Base DF has been prov'd equal to the Base FC. Therefore the Angle FAD is † equal to the Angle FBC: Moreover, AG has been prov'd equal to BH; but FB, also, is equal to AF; therefore the two Sides FA, AG, are equal to the two Sides FB, BH; and the Angle FAG is equal to the Angle FBH, as has been demonstrated; wherefore the Base GF is † equal to the Base FH. Again, because GE has been prov'd equal to EH, and EF is common, the two Sides GE, EF, are equal to the two Sides HE, EF; but the Base HF is equal to the Base FG; therefore the Angle GEF is † equal to the Angle HEF; and so both the Angles GEF, HEF, are right Angles: Therefore FE makes right Angles with GH, which is any-how drawn thro' E. After the same manner we demonstrate, that FE is at right Angles to all right Lines that are drawn in the Plane to it; but a right Line is * at right Angles to a Plane, when it is at right Angles to all right Lines drawn to it in the Plane. Therefore FE is at right Angles to a Plane drawn thro' the right Line A

* 15. 1.

† 4. 1.

† 6. 1.

† 8. 5.

* Def. 3. of this.

E, DE, right Line e Point F H, FB: are equal tain * the shall be + ED equal the Angle ngle AEG re AGE, gles of the ch to each, viz. those other Sides des of the nd AG to E is com- shall be + , likewise, use AD is Sides FA, BC, each 'd equal to is † equal een prov'd F; there- to the two equal to the wherefore Again, be- and EF is equal to the equal to the † equal to gles GEF, makes right wn thro' E. that FE is re drawn in ight Angles to all right re FE is at right Lines AB,

AB, CD. Wherefore, *if to two right Lines, cutting one another, a third stands at right Angles in the common Section, it shall be also at right Angles to the Plane drawn thro' the said Lines; which was to be demonstrated.*

PROPOSITION V.

THEOREM.

If to three right Lines, touching one another, a third stands at right Angles in their common Section, those three right Lines shall be in one and the same Plane.

LET the right Line AB stand at right Angles, in the Point of Contact B, to the three right Lines BC, BD, BE. I say, BC, BD, BE, are in one and the same Plane.

For, if they are not, let BD, BE, be in one Plane, and BC above it; and let the Plane passing thro' AB, BC, be produced, and it will * make the common * ^{3 of this.} Section, with the other Plane, a strait Line, which let be BF; then three right Lines AB, BC, BF, are in one Plane drawn thro' AB, BC: And since AB stands at right Angles to BD and BE, it shall be † at † ^{4 of this.} right Angles to a Plane drawn thro' BE, DB; and so AB shall make ‡ right Angles with all right Lines † ^{Def. 3.} touching it that are in the same Plane: But BF being in the said Plane, touches it; wherefore the Angle ABF is a right Angle: But the Angle ABC (by the Hyp.) is also a right Angle; therefore the Angle ABF is equal to the Angle ABC, and they are both in the same Plane, which cannot be; and so the right Line BC is not above the Plane passing thro' BE and BD. Wherefore the three Lines BC, BD, BE, are in one and the same Plane. Therefore, *if to three right Lines, touching one another, a third stands at right Angles in their common Section, those three right Lines shall be in one and the same Plane; which was to be demonstrated.*

PROPOSITION VI.

THEOREM.

If two right Lines be perpendicular to one and the same Plane, those right Lines are parallel one another.

LET two right Lines, AB, CD, be perpendicular to one and the same Plane. I say, AB is parallel to CD.

For, let them meet the Plane in the Points B, and join the right Line BD, to which let DE be drawn in the same Plane, at right Angles; make DE equal to AB; and join BE, AE, AD.

* Def. 3.
of this.

Then, because AB is at right Angles to the said Plane, it shall be * at right Angles to all right Lines, touching it, drawn in the Plane; but AB touches BD, BE, which are in the said Plane; therefore each of the Angles ABD, ABE, is a right Angle. So, by the same Reason, likewise, is each of the Angles CDB, CDE, a right Angle. Then, because AB is equal to DE, and BD is common, the two Sides AB, BD, shall be equal to the two Sides ED, BD, but they contain right Angles: Therefore the Line AD is † equal to the Base BE. Again, because AB is equal to DE, and AD to BE; the two Sides AB, BE, are equal to the two Sides ED, DA; but AD is their Base, is common; wherefore the Angle ABE is † equal to the Angle EDA. But ABE is a right Angle; therefore EDA is also a right Angle; so ED is perpendicular to DA: But it is also perpendicular to BD and DC; therefore ED is at right Angles, in the Point of Contact, to three right Lines BD, DA, DC: Wherefore these three last right

† 4. 1.

† 8. 1.

* 5 of this.

† 2 of this.

† 28. 1.

Lines are * in one Plane. But BD, DA, are in the same Plane as AB is; for every Triangle is † in the same Plane; therefore it is necessary, that AB, BD, DC, be in one Plane. But both the Angles ABE, BDC, are right Angles; wherefore AB is † parallel to CD. Therefore, *if two right Lines be perpendicular to one and the same Plane, those right Lines are parallel one another; which was to be demonstrated.*

VI.

PROPOSITION VII.

THEOREM.

If there be two parallel Lines, and any Point be taken in both of them, the right Lines joining those Points shall be in the same Planes as the Parallels are.

LET AB, CD, be two parallel right Lines, in which are taken any Points E, F. I say, a right Line joining the Points E, F, is in the same Plane as the Parallels are.

For, if it be not, let it be elevated above the same, if possible, as EGF, thro' which let some Plane be drawn, whose Section, with the Plane in which the Parallels are, let * be the right Line EF; then the * ^{3 of this.} two right Lines EGF, EF, will include a Space, which is † absurd: Therefore a right Line, drawn † ^{Axiom} from the Point E to the Point F, is not elevated ^{10. 1.} above the Plane; and, consequently, it must be in that passing thro' the Parallels AB, CD. Wherefore, *if there be two parallel Lines, and any Points be taken in both of them, the right Line joining these Points shall be in the same Plane as the Parallels are; which was to be demonstrated.*

PROPOSITION VIII.

THEOREM.

If there be two parallel right Lines, one of which is perpendicular to some Plane; then shall the other be perpendicular to the same Plane.

LET AB, CD, be two parallel right Lines, one ^{See the Fig.} of which, as AB, is perpendicular to some Plane. ^{of Prop. VI.} I say, the other, CD, is also perpendicular to the same Plane.

For, let AB, CD, meet the Plane in the Points B, D, and let BD be join'd; then AB, CD, BD, are * in one Plane. Let DE be drawn in the Plane, at * ^{7 of this,} right Angles to BD, and make DE equal to AB, and

- and join BE, AE, AD : Then, since AB is perpendicular to the Plane, it will * be perpendicular to all right Lines, touching it, drawn in the same Plane ; therefore each of the Angles ABD, ABE, is a right Angle. And since the right Line BD falls on the right Lines AB, CD ; the Angles ABD, CDB, shall be † equal to two right Angles : Therefore the Angle CDB is also a right Angle ; and so CD is perpendicular to DB. And since AB is equal to DE, and BD is common ; the two Sides AB, BD, are equal to the two Sides ED, DB. But the Angle ABD is equal to the Angle EDB, for each of them is a right Angle ; therefore the Base AD is † equal to the Base BE. Again, since AB is equal to DE, and BE to AD, the two Sides AB, BE, shall be equal to the two Sides ED, DA, each to each : But the Base AE is common ; wherefore the Angle ABE is * equal to the Angle EDA : But the Angle ABE is a right Angle, therefore EDA is also a right Angle, and ED is perpendicular to DA : But it is likewise perpendicular to DB ; therefore ED shall also be † perpendicular to the Plane passing thro' BD, DA, and, likewise, shall be † at right Angles to all right Lines, drawn in the said Plane, that touch it. But DC is in the Plane passing thro' BD, DA, because AB, BD, are * in that Plane ; and DC is † in the same Plane that AB and BD are in ; wherefore ED is at right Angles to DC, and so CD is at right Angles to DE, as also to DB : Therefore CD stands at right Angles, in the common Section D, to two right Lines DE, DB, mutually cutting one another ; and, accordingly, is at right Angles to the Plane passing thro' DE, DB ; which was to be demonstrated.
- * Def. 3.
- † 29. 1.
- † 4. 1.
- * 8. 1.
- † 4 of this.
- † Def. 3.
- * 2 of this.
- † 7 of this.

PROPOSITION IX.

THEOREM.

Right Lines that are parallel to the same right Line, not being in the same Plane with it, are also parallel to each other.

LET both the right Lines AB, CD, be parallel to the right Line EF, not being in the same Plane with it. I say, AB is parallel to CD.

For, assume any Point G in EF, from which Point G let GH be drawn, at right Angles to EF, in the Plane passing thro' EF, AB: Also, let GK be drawn at right Angles to EF in the Plane passing thro' EF, CD: Then, because EF is perpendicular to GH, and GK, the Line EF shall also be * at right Angles to a * 4 of this. Plane passing thro' GH, GK: But EF is parallel to AB; therefore AB is † also at right Angles to the Plane † 8 of this. passing thro' HGK. For the same Reason, CD is also at right Angles to the Plane passing thro' HGK; and therefore AB, and CD, will be both at right Angles to the Plane passing thro' HGK. But if two right Lines be at right Angles to the same Plane, they shall be † parallel to each other; therefore AB is parallel † 6 of this. to CD; which was to be demonstrated.

PROPOSITION X.

PROBLEM.

If two right Lines, touching one another, be parallel to two other right Lines, touching one another, but not in the same Plane, those right Lines contain equal Angles.

LET two right Lines AB, BC, touching one another, be parallel to two right Lines DE, EF, touching one another, but not in the same Plane. I say, the Angle ABC is equal to the Angle DEF.

For, take BA, BC, ED, EF, equal to one another, and join AD, CF, BE, AC, DF: Then, because BA is equal and parallel to ED, the right Line

- * 30. I. AD shall also be * equal and parallel to BE. For the same Reason, CF will be equal and parallel to BE; therefore AD, CF, are both equal and parallel to BE. But right Lines that are parallel to the same right Line, not being in the same Plane with it, will
- † 9 of this. be † parallel to each other. Therefore AD is parallel and equal to CF; but AC, DF, joins them;
- † 33. I. wherefore AC is † equal and parallel to DF. And because two right Lines AB, BC, are equal to two right Lines DE, EF, and the Base AC equal to the Base DF, the Angle ABC will be * equal to the Angle DEF. Therefore, *if two right Lines, touching one another, be parallel to two other right Lines, touching one another, but not in the same Plane, those right Lines contain equal Angles; which was to be demonstrated.*
- * 8. I.

PROPOSITION XI.

PROBLEM.

From a Point given above a Plane, to draw a right Line perpendicular to that Plane.

LET A be a Point given, above the given Plane BH. It is requir'd to draw a right Line from the Point A, perpendicular to the Plane BH.

- * 12. I. Let a right Line BC be any-how drawn in the Plane BH; and let AD be drawn * from the Point A perpendicular to BC; then, if AD be perpendicular to the Plane BH, the Thing requir'd is already done; but, if not, let DE be drawn in the Plane from the Point D, at right Angles to BC; and let AF be drawn * from the Point A perpendicular to DE. Lastly, thro' F draw GH, parallel to BC.

- † 4 of this. Then, because BC is perpendicular to both DA and DE, BC will also be † perpendicular to a Plane passing thro' ED, DA. But GH is parallel to BC; and if there are two right Lines parallel, one of which is at right Angles to some Plane, then shall
- † 8 of this. the other be † at right Angles to the same Plane: Wherefore GH is at right Angles to the Plane passing thro' ED, DA; and so is * perpendicular to all the right Lines, in the same Plane, that touch it. But AF, which
- * Def. 3.

which is in the Plane passing thro' ED and DA, doth touch it. Therefore GH is perpendicular to AF, and so AF is perpendicular to GH; but AF, likewise, is perpendicular to DE; therefore AF is perpendicular to both HG, DE. But if a right Line stands at right Angles to two right Lines, in their common Section, that Line will be † at right Angles to the † 14 of this. Plane passing thro' these Lines. Therefore AF is perpendicular to the Plane drawn thro' ED, GH; that is, to the given Plane BH. Therefore AF is drawn from the given Point A, above the given Plane BH, perpendicular to the said Plane; *which was to be done.*

PROPOSITION XII,

PROBLEM.

To erect a right Line perpendicular to a given Plane, from a Point given therein.

LET A be a given Point in a given Plane MN. It is requir'd to draw a right Line from the Point A, at right Angles to the Plane MN.

Let some Point B be supposed above the given Plane, from which let BC be drawn * perpendicular * 11 of this. to the said Plane; and let AD be drawn † from A parallel to BC. † 31. 1.

Then, because AD, CB, are two parallel right Lines, one of which, viz. BC, is perpendicular to the Plane MN; the other, AD, shall be † also perpendicular to the same Plane. † 8 of this. Therefore, *a right Line is erected perpendicular to a given Plane, from a Point given therein; which was to be done.*

PROPOSITION XIII.

THEOREM.

Two right Lines cannot be erected at right Angles, to a given Plane, from a Point therein given.

FOR, if it is possible, let two right Lines AB, AC, be erected perpendicular to a given Plane on the same Side, at a given Point A, in a given Plane.

* 3 of this.

† Def. 3.

Then let a Plane be drawn thro' BA, AC, cutting the given Plane thro' A in the right Line * DAE; therefore the right Lines AB, AC, DAE, are in one Plane: And because CA is perpendicular to the given Plane, it shall also be † perpendicular to all right Lines drawn in that Plane, and touching it: But DAE, being in the given Plane, touches it; therefore the Angle CAE is a right Angle. For the same Reason, BAE is also a right Angle; wherefore the Angle CAE is equal to BAE, and they are both in one Plane, which is absurd. Therefore, *two right Lines cannot be erected at right Angles, to a given Plane, from a Point therein given; which was to be demonstrated.*

PROPOSITION XIV.

THEOREM.

Those Planes, to which the same right Line is perpendicular, are parallel to each other.

LET the right Line AB be perpendicular to each of the Planes CD, EF. I say, these Planes are parallel.

For, if they be not, let them be produced till they meet each other, and let the right Line GH be the common Section, in which take any Point K, and join AK, BK. Then, because AB is perpendicular to the Plane EF, it shall also be perpendicular to the right Line BK, being in the Plane EF produced; wherefore the Angle ABK is a right Angle. And, for the same Reason, BAK is also a right Angle.

And

And so the two Angles ABK , BAK , of the Triangle ABK , are equal to two right Angles, which is * im- * 17. 1. possible: Therefore the Planes CD , EF , being produced, will not meet each other, and so are necessarily parallel. Therefore, *those Planes, to which the same right Line is perpendicular, are parallel to each other*; which was to be demonstrated.

PROPOSITION XV.

THEOREM.

If two right Lines, touching one another, be parallel to two right Lines, touching one another, and not being in the same Plane with them, the Planes drawn thro' those right Lines are parallel to each other.

LET two right Lines AB , BC , touching one another, be parallel to two right Lines DE , EF , touching one another, but not in the same Plane with them. I say, the Planes passing thro' AB , BC , and DE , EF , being produced, will not meet each other.

For, let BG be drawn, from the Point B , perpendicular to the Plane passing thro' DE , EF , meeting the same in the Point G ; and thro' G let GH be drawn parallel to ED , and GK parallel to EF ; then, because BG is perpendicular to the Plane passing thro' DE , EF , it shall also make * right Angles * Def. 3. with all right Lines that touch it, and are in the same Plane: But GH and GK , which are both in the same Plane, touch it; therefore each of the Angles BGH , BGK , is a right Angle. And since BA is parallel to GH , the Angles GBA , GBH , are † † 29. 1. equal to the right Angles: But BGH is a right Angle: Wherefore GBA shall also be a right Angle, and so BG is perpendicular to BA . For the same Reason, GB is also perpendicular to BC ; therefore, since a right Line BG stands at right Angles to two right Lines BA , BC , mutually cutting each other; BG shall also be † at right Angles to the Plane drawn † 4 of this. thro' BA , BC . But it is perpendicular to the Plane drawn thro' DE , EF ; therefore BG is perpendicular to both the Planes drawn thro' AB , BC , and DE ,

DE, EF. But those Planes to which the same right Line is perpendicular, are * parallel; therefore the Plane drawn thro' AB, BC, is parallel to the Plane drawn thro' DE, EF. Wherefore, *if two right Lines, touching one another, be parallel to two right Lines, touching one another, and not being in the same Plane with them, the Planes drawn thro' these right Lines are parallel to each other.*

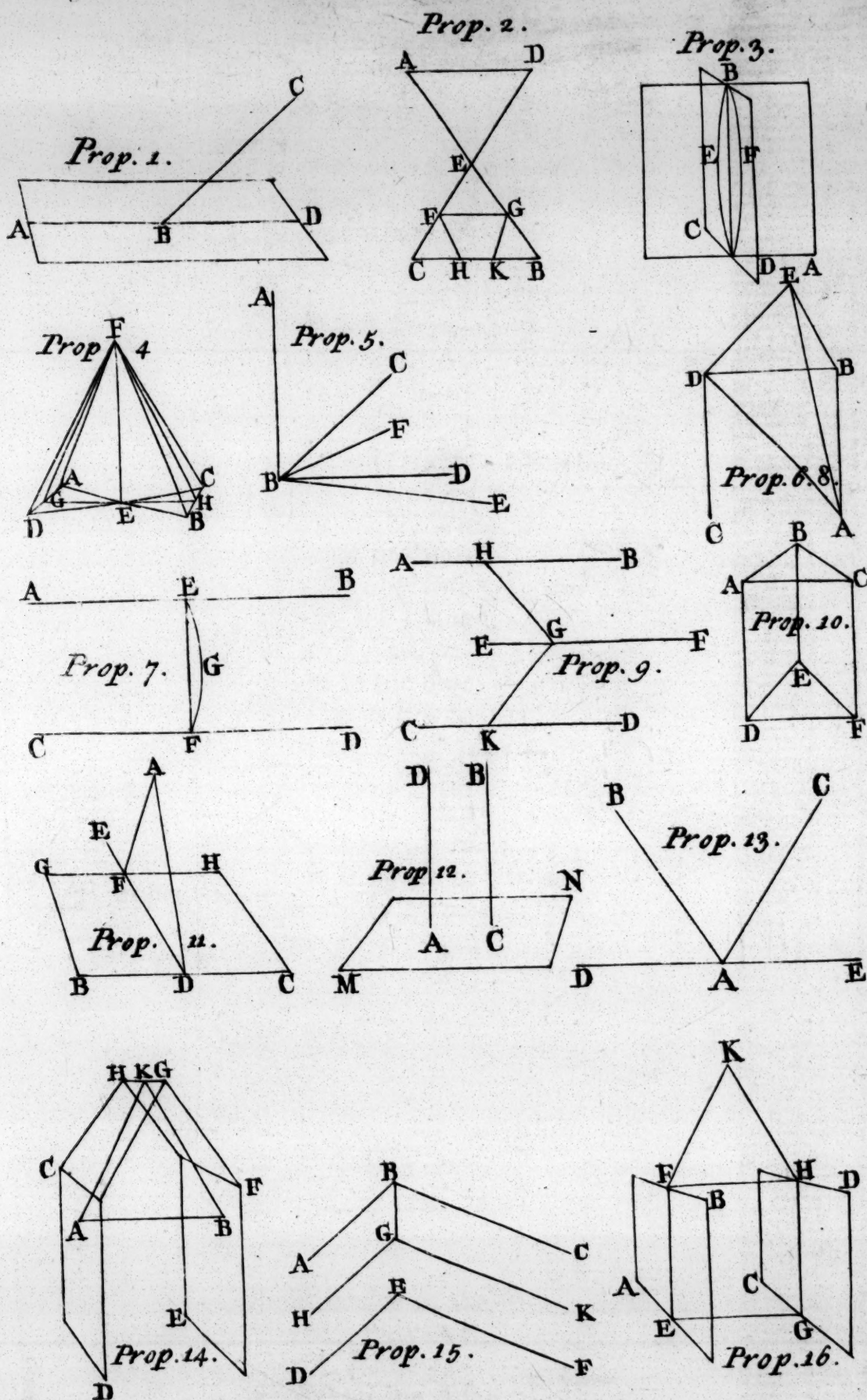
PROPOSITION XVI.

THEOREM.

If two parallel Planes are cut by any other Plane, their common Sections will be parallel.

LET two parallel Planes, AB, CD, be cut by any Plane EFHG; and let their common Sections be EF, GH. I say, EF is parallel to GH.

For, if it is not parallel, EF, GH, being produced, will meet each other either on the Side FH, or EG. First, let them be produced on the Side FH, and meet in K; then, because EFK is in the Plane AB, all Points taking in EFK will be in the same Plane. But K is one of the Points that is in EFK; therefore K is in the same Plane AB. For the same Reason, K is also in the Plane CD; wherefore the Planes AB, CD, will meet each other. But they do not meet, since they are suppos'd parallel; therefore the right Lines EF, GH, will not meet on the Side FH. After the same manner it is prov'd, that they will not meet, if produced, on the Side EG. But right Lines, that will neither way meet each other, are parallel; therefore EF is parallel to GH. *If, therefore, two parallel Planes are cut by any other Plane, their common Sections will be parallel; which was to be demonstrated.*



1

PROPOSITION XVII.

THEOREM.

If two right Lines are cut by parallel Planes, they shall be cut in the same Proportion.

LET two right Lines AB , CD , be cut by parallel Planes GH , KL , MN , in the Points A , E , B , C , F , D . I say, as the right Line AE is to the right Line EB , so is CF to FD .

For, let AC , BD , AD , be join'd; let AD meet the Plane KL in the Point X ; and join EX , XF . Then, because two parallel Planes KL , MN , are cut by the Plane $EBDX$, their common Sections EX , BD , are * parallel. For the same Reason, because * 16 of this. two parallel Planes GH , KL , are cut by the Plane $AXFC$, their common Sections AC , FX , are parallel; and because EX is drawn parallel to the Side BD of the Triangle ABD , it shall be, as AE is to EB , so is $\dagger AX$ to XD . Again, because XF is $\dagger$ 2. 6. drawn parallel to the Side AC of the Triangle ADC , it shall be $\dagger$, as AX to XD , so is CF to FD . But it has been prov'd, as AX is to XD , so is AE to EB . Therefore, as AE is to EB , so is $\dagger CF$ to $\dagger$ 11. 5. FD . Wherefore, *if two right Lines are cut by parallel Planes, they shall be cut in the same Proportion; which was to be demonstrated.*

PROPOSITION XVIII.

THEOREM.

If a right Line be perpendicular to some Plane, then all Planes passing thro' that Line will be perpendicular to the same Plane.

LET the right Line AB be perpendicular to the Plane CL . I say, all Planes that pass thro' AB , are likewise perpendicular to the Plane CL .

For, let a Plane DE pass thro' the right Line AB , whose common Section, with the Plane CL , is the right Line CE ; and take some Point F in CE ; from which

* Def. 3.

† 8 of this.

† Def. 4. of this.

which let FG be drawn in the Plane DE , perpendicular to the right Line CE : Then, because AB is perpendicular to the Plane CL , it shall also be * perpendicular to the right Lines which touch it, and are in the same Plane: Wherefore it is perpendicular to CE , and, consequently, the Angle ABF is a right Angle: But GFB is likewise a right Angle; therefore AB is parallel to FG . But AB is at right Angles to the Plane CL ; therefore FG will be † at right Angles to that same Plane. But one Plane is perpendicular to another, when the right Lines, drawn in one of the Planes perpendicular to the common Section of the Planes, are ‡ perpendicular to the other Plane. But FG is drawn in one Plane DE , perpendicular to the common Section CE of the Planes, and it has been prov'd to be perpendicular to the Plane CL ; therefore the Plane DE is at right Angles to the Plane CL . After the same manner it is demonstrated, that all Planes, passing thro' the right Line AB , are perpendicular to the Plane CL . Therefore, *if a right Line be perpendicular to some Plane, then all Planes, passing thro' that Line, will be perpendicular to the same Plane*; which was to be demonstrated.

PROPOSITION XIX.

THEOREM.

If two Planes, cutting each other, be perpendicular to some Plane, then their common Section will be perpendicular to that same Plane.

LET two Planes AB , BC , cutting each other, be perpendicular to some third Plane, and let their common Section be BD . I say, BD is perpendicular to the said third Plane, which let be ADC .

For, if possible, let BD not be perpendicular to the third Plane; and from the Point D let DE be drawn in the Plane AB , perpendicular to AD ; and let DF be drawn, in the Plane BC , perpendicular to CD : Then, because the Plane AB is perpendicular to the third Plane, and DE is drawn in the Plane AB , perpendicular to their common Section AD , DE

shall be * perpendicular to the third Plane. In like* *Def. 4.*
 manner we prove, that DF also is perpendicular to
 the said Plane; wherefore two right Lines stand at
 right Angles, to this third Plane, on the same Side,
 at the same Point D; which is † absurd: Therefore, † *13 of this.*
 to this third Plane cannot be erected any right Lines
 perpendicular at D, and on the same Side, except
 BD, the common Section of the Planes AB, BC:
 Wherefore DB is perpendicular to the third Plane.
If, therefore, two Planes, cutting each other, be per-
pendicular to some Plane, then their common Section
will be perpendicular to that same Plane; which was
to be demonstrated.

PROPOSITION XX.

THEOREM.

*If a solid Angle be contained under three plain
 Angles, any two of them, howsoever taken,
 are greater than the third.*

LET the solid Angle A be contain'd under three
 plain Angles BAC, CAD, DAB. I say, any
 two of the Angles BAC, CAD, DAB, are greater
 than the third, howsoever taken.

For, if the Angles BAC, CAD, DAB, be equal,
 it is evident, that any two, howsoever taken, are
 greater than the third; but, if not, let BAC be the
 greater, and make * the Angle BAE, at the Point A, * *23. 1.*
 with the right Line AB, in a Plane passing thro' BA,
 AC, equal to the Angle DAB; make AE equal to
 AD; thro' E draw BEC, cutting the right Lines
 AB, AC, in the Points B, C; and join DB, DC:
 Then, because DA is equal to AE, and AB is com-
 mon, the two Sides DA, AB, are equal to the two
 Sides AE, AB; but the Angle DAB is equal to the
 Angle BAE; therefore the Base DB is † equal to † *4. 1.*
 the Base BE: And since the two Sides DB, DC,
 are greater than BC, and DB has been prov'd equal
 to BE, the remaining Side DC shall be greater than
 the remaining Side EC; and since DA is equal to
 AE, and AC is common, and the Base DC greater
 than the Base EC, the Angle DAC shall be † greater † *25. 1.*
 than

than the Angle EAC. But, from Construction, the Angle DAB is equal to the Angle BAE; wherefore the Angles DAB, DAC, are greater than the Angle BAC. After this manner we demonstrate, if any two other Articles be taken, that they are greater than the third. Therefore, *if a solid Angle be contained under three plain Angles, any two of them, howsoever taken, are greater than the third; which was to be demonstrated.*

PROPOSITION XXI.

THEOREM.

Every solid Angle is contained under plain Angles together, less than four right ones.

LET A be a solid Angle, contain'd under plain Angles BAC, CAD, DAB. I say, the Angles BAC, CAD, DAB, are less than four right Angles.

For, take any Points B, C, D, in each of the Lines AB, AC, AD; and join BC, CD, DB: Then, because the solid Angle at B is contain'd under three plain Angles CBA, ABD, CBD, any two of these ^{* 20 of this.} are * greater than the third: Therefore the Angles CBA, ABD, are greater than the Angle CBD. For the same Reason, the Angles BCA, ACD, are greater than the Angle BCD; and the Angles CDA, ADB, greater than the Angle CDB. Wherefore the six Angles CBA, ABD, BCA, ACD, ADC, ADB, are greater than the three Angles CBD, BCD, CDB. But the three Angles CBD, BCD, CDB, are † equal to two right Angles; wherefore the six Angles CBA, ABD, BCA, ACD, ADC, ADB, are greater than two right Angles. And since the three Angles of each of the Triangles ABC, ACD, ADB, are equal to two right Angles, the nine Angles of those Triangles CBA, BCA, BAC, ACD, CAD, ADC, ADB, ABD, DAB, are equal to six right Angles; six of which Angles, CBA, BCA, ACD, ADC, ADB, ABD, are greater than two right Angles. Therefore the three other Angles BAC, CAD, DAB, which contain the solid Angle,

† 32. 1.

gle, will be less than four right Angles. Wherefore, every solid Angle is contained under Angles, together, less than four plain right ones; which was to be demonstrated.

PROPOSITION XXII.

THEOREM.

If there be three plain Angles, whereof two, any-how taken, are greater than the third, and the right Lines that contain them be equal; then it is possible to make a Triangle of the right Lines joining the equal right Lines which form the Angles.

LET ABC, DEF, GHK, be given plain Angles, any two whereof are greater than the third; and let the equal right Lines AB, BC, DE, EF, GH, HK, contain them; and let AC, DF, GK, be join'd. I say, it is possible to make a Triangle of AC, DF, GK; that is, any two of them, howsoever taken, are greater than the third.

For, if the Angles at B, E, H, are equal; then AC, DF, GK, will be * equal, and any two of them * 4. 1. greater than the third; but, if not, let the Angles at B, E, H, be unequal; and let the Angle B be greater than either of the others at E, or H: Then the right Line AC will be † greater than either DF, or GK; † 24. 1. and it is manifest, that AC, together with either DF, or GK, is greater than the other. I say, likewise, that DF, GK, together, are greater than AC. For make †, † 23. 1. at the Point B, with the right Line AB, the Angle ABL equal to the Angle GHK; and make BL equal to either AB, BC, DE, EF, GH, HK; and join AL, CL. Then, because the two Sides AB, BL, are equal to the two Sides GH, HK, each to each, and they contain equal Angles, the Base AL shall be equal to the Base GK. And since the Angles E and H are greater than the Angle ABC, whereof the Angle GHK is equal to the Angle ABL, the other Angle at E shall be greater than the Angle LBC. And since the two Sides LB, BC, are equal to the two Sides DE, EF, each to each, and the Angle DEF

* 24. 1.

DEF is greater than the Angle LBC, the Base DF shall be * greater than the Base LC. But GK has been prov'd equal to AL; therefore DF, GK, are greater than AL, LC: But AL, LC, are greater than AC; wherefore DF, GK, shall be much greater than AC. Therefore any two of the right Lines AC, DF, GK, howsoever taken, are greater than the other; and so a Triangle may be made of AC, DF, GK; *which was to be demonstrated.*

PROPOSITION XXIII.

PROBLEM.

To make a solid Angle of three plain Angles, whereof any two, howsoever taken, are greater than the third; but these three Angles must be less than four right Angles.

LET ABC, DEF, GHK, be three plain Angles given, whereof any two, howsoever taken, are greater than the other; and let the said three Angles be less than four right Angles. It is requir'd to make a solid Angle of three plain Angles equal to ABC, DEF, GHK.

* 22. of this.

† 22. 1.

† 5. 4.

Let the right Lines AB, BC, DE, EF, GH, HK, be cut off equal; and join AC, DF, GK; then it is possible to make * a Triangle of three right Lines equal to AC, DF, GK: And so let † the Triangle LMN be made, so that AC be equal to LM, and DF to MN, and GK to LN; and let the Circle LMN be describ'd ‡ about the Triangle, whose Centre let be X, which will be either within the Triangle LMN, or on one Side thereof, or without the same.

* 8. 1.

First, Let it be within, and join LX, MX, NX: I say, AB is greater than LX. For, if this be not so, AB shall be either equal to LX, or less. First, let it be equal; then, because AB is equal to LX, and also to BC, LX shall be equal to BC: But LX is equal to XM; therefore the two Sides AB, BC, are equal to the two Sides LX, XM, each to each. But the Base AC is put equal to the Base LM; wherefore the Angle ABC shall be * equal to the Angle LXM.

L X M. For the same Reason, the Angle D E F is equal to the Angle M X N, and the Angle G H K to the Angle N X L; therefore the three Angles A B C, D E F, G H K, are equal to the three Angles L X M, M X N, N X L. But the three Angles L X M, M X N, N X L, are * equal to four right Angles; * Cor. 15. 1. and so the three Angles A B C, D E F, G H K, shall also be equal to four right Angles: But they are put less than four right Angles, which is absurd; therefore A B is not equal to L X. I say, also, it is neither less than L X; for, if this be possible, make X O equal to A B, and X P to B C, and join O P: Then, because A B is equal to B C, X O shall be equal to X P; and the remaining Part O L equal to the remaining Part P M; and so L M is † parallel to O P, and the † 2. 6. Triangle L M X is equiangular to the Triangle O P X: Wherefore X L is † to L M, as X O is to O P; and † 4. 6. (by Alternation) as X L is to X O, so is L M to O P. But L X is greater than X O; therefore L M shall also be greater than O P. But L M is put equal to A C; wherefore A C shall be greater than O P: And so, because the two right Lines A B, B C, are equal to the two right Lines O X, X P, and the Base A C greater than the Base O P; the Angle A B C will be * greater than the Angle O X P. In like * 25. 1. manner we demonstrate, that the Angle D E F is greater than the Angle M X N, and the Angle G H K than the Angle N X L; therefore the three Angles A B C, D E F, G H K, are greater than the three Angles L X M, M X N, N X L: But the Angles A B C, D E F, G H K, are put less than four right Angles; therefore the Angles L X M, M X N, N X L, shall be less by much than four right Angles, and also equal † to four right Angles, which is absurd: † Cor. 15. 1. Wherefore A B is not less than L X. It has also been prov'd not to be equal to it; therefore it must necessarily be greater. On the Point X raise † X R, † 12 of this. perpendicular to the Plane of the Circle L M N, whose Length let be such, that the Square thereof be equal to the Excess by which the Square of A B exceeds the Square of L X; and let R L, R M, R N, be join'd: Because R X is perpendicular to the Plane of the Circle L M N, it shall also be * perpendicular * Def. 3. to L X, M X, N X: And because L X is equal to

* 4. I.

† 47. I.

‡ S. I.

* 20. I.

XM , and XR is common, and at right Angles to them, the Base LR shall be * equal to the Base RM . For the same Reason, RN is equal to RL , or RM ; therefore three right Lines RL , RM , RN , are equal to each other. And because the Square of XR is equal to the Excess by which the Square of AB exceeds the Square of LX , the Square of AB will be equal to the Squares of LX , XR , together: But the Square of RL is † equal to the Squares of LX , XR , for LXR is a right Angle; therefore the Square of AB will be equal to the Square of RL ; and so AB is equal to RL . But BC , DE , EF , GH , HK , are every of them equal to AB ; and RN , or RM , equal to RL ; wherefore AB , BC , DE , EF , GH , HK , are each equal to RL , RM , or RN : And since the two Sides RL , RM , are equal to the two Sides AB , BC ; and the Base LM is put equal to the Base AC ; the Angle LRM shall be ‡ equal to the Angle ABC . For the same Reason, the Angle MRN is equal to the Angle DEF , and the Angle LRN equal to the Angle $G HK$; therefore a solid Angle is made at R of three plain Angles LRM , MRN , LRN , equal to three plain Angles given, ABC , DEF , $G HK$.

Now, let the Centre of the Circle X be in one Side of the Triangle, viz. in the Side MN , and join XL . I say, again, that AB is greater than LX . For, if it be not so, AB will be either equal, or less than LX . First, let it be equal; then the two Sides AB , BC , are equal to the two Sides MX , LX , that is, they are equal to MN : But MN is put equal to DF ; therefore DE , EF , are equal to DF , which is * impossible; therefore AB is not equal to LX . In like manner we prove, that it is neither lesser; for the Absurdity will much more evidently follow: Therefore AB is greater than LX . And if, in like manner as before, the Square of RX be made equal to the Excess by which the Square of AB exceeds the Square of LX , and RX be raised at right Angles to the Plane of the Circle, the Problem will be done.

Lastly, Let the Centre X of the Circle be without the Triangle LMN , and join LX , MX , NX : I say, AB is greater than LX . For, if it be not, it must

must either be equal, or less. First, let it be equal ;
then the two Sides AB , BC , are equal to the two
Sides MX , XL , each to each ; and the Base AC is
equal to the Base ML ; therefore the Angle ABC is
equal to the Angle MXL . For the same Reason,
the Angle GHK is equal to the Angle LXN ; and
so the whole Angle MXN is equal to the two An-
gles ABC , GHK : But the Angles ABC , GHK ,
are greater than the Angle DEF ; therefore the
Angle MXN is greater than DEF : But because the
two Sides DE , EF , are equal to the two Sides MX ,
 XN , and the Base DF is equal to the Base MN ; the
Angle MXN shall be equal to the Angle DEF : But
it has been prov'd greater, which is absurd ; there-
fore AB is not equal to LX . Moreover, we will
prove, that it is not less ; wherefore it shall be neces-
sarily greater. And if, again, XR be rais'd at right
Angles to the Plane of the Circle, and made equal to
the Side of that Square by which the Square of AB
exceeds the Square of LX ; the Problem will be de-
termin'd. Now, I say, AB is not less than LX :
For, if it is possible, that it can be less, make XO equal
to AB , and XP equal to BC , and join OP ; then,
because AB is equal to BC , XO shall be equal to
 XP , and the remaining Part OL equal to the
remaining Part PM ; therefore LM is * parallel to * 2. 6.
 PO , and the Triangle LMX equiangular to the
Triangle PXO : Wherefore, as XL is to LM , so † 4. 6.
is XO to OP ; and (by Alternation) as LX is to
 XO , so is LM to OP : But LX is greater than XO ;
therefore LM is greater than OP ; but LM is equal
to AC ; wherefore AC shall be greater than OP ;
and so, because the two Sides AB , BC , are equal to
the two Sides OX , XP , each to each ; and the Base
 AC is greater than the Base OP ; the Angle ABC
shall be ‡ greater than the Angle OMP . So, likewise, ‡ 25. 1.
if XR be taken equal to XO or XP , and OR be
join'd, we prove, that the Angle GHK is greater
than the Angle ORX . At the Point X , with the
right Line LX , make the Angle LXS equal to the
Angle ABC , and the Angle LXT equal to the An-
gle GHK , and XS , XT , each equal to XO , and
join OS , OT , ST ; then, because the two Sides
 AB , BC , are equal to the two Sides OX , XS , and

the Angle ABC is equal to the Angle OXS , the Base AC , that is, LM , shall be equal to the Base OS . For the same Reason, LN is also equal to OT : And since the two Sides ML , LN , are equal to the two Sides OS , OT ; and the Angle MLN greater than the Angle SOT ; the Base MN shall be greater than the Base ST . But MN is equal to DF ; therefore DF shall be greater than ST . Wherefore, because the two Sides DE , EF , are equal to the two Sides SX , XT ; and the Base DF is greater than the Base ST ; the Angle DEF shall be greater than the Angle SXT . But the Angle SXT is equal to the Angles ABC , GHK ; therefore the Angle DEF is greater than the Angles ABC , GHK : But it is also less, which is absurd; *which was to be demonstrated.*

PROPOSITION XXIV.

THEOREM.

If a Solid be contained under six parallel Planes, the opposite Planes thereof are equal Parallelograms.

LET the Solid $CDGH$ be contain'd under parallel Planes AC , GF , BG , CE , FB , AE . I say, the opposite Planes thereof are equal Parallelograms.

For, because the parallel Planes BG , CE , are cut * 16 of this. by the Plane AC , their common Sections are * parallel; wherefore AB is parallel to CD . Again, because the two parallel Planes BF , AE , are cut by the Plane AC , their common Sections are parallel; therefore AD is parallel to BC . But AB has been prov'd to be parallel to CD ; wherefore AC shall be a Parallelogram. After the same manner we demonstrate, that CE , FG , GB , BF , or AE , is a Parallelogram.

Let AH , DF , be join'd: Then, because AB is parallel to DC , and BH to CF ; the Lines AB , BH , touching each other, shall be parallel to the Lines DC , CF , touching each other, and not being in the † 10 of this. same Plane; wherefore they shall † contain equal Angles: And so the Angle ABH is equal to the Angle DCF .

DCF. And since the two Sides AB, BH, are $\dagger$ equal $\dagger$ 34. 1. to the two Sides DC, CF, and the Angle ABH equal to the Angle DCF; the Base AH shall be $*$ equal to $*$ 4. 1. the Base DF, and the Triangle ABH equal to the Triangle DCF: And since the Parallelogram BG is $\dagger$ double to the Triangle ABH, and the Parallelogram $\dagger$ 41. 1. CE to the Triangle DCF; the Parallelogram BG shall be equal to the Parallelogram CE. In like manner we demonstrate, that the Parallelogram AC is equal to the Parallelogram GF, and the Parallelogram AE equal to the Parallelogram BF. *If, therefore, a Solid be contained under six parallel Planes, the opposite Planes thereof are equal Parallelograms; which was to be demonstrated.*

Coroll. It follows, from what has been now demonstrated, that, if a Solid be contain'd under six parallel Planes, the opposite Planes thereof are similar and equal, because each of the Angles are equal, and the Sides about the equal Angles are proportional.

PROPOSITION XXV.

THEOREM.

If a solid Parallelopipedon be cut by a Plane, parallel to opposite Planes; then, as Base is to Base, so shall Solid be to Solid.

LET the solid Parallelopipedon ABCD be cut by a Plane YE, parallel to the opposite Planes RA, DH. I say, as the Base EF Φ A is to the Base EHCF, so is the Solid ABFY to the Solid EGCD.

For, let AH be both ways produced, and make HM, MN, &c. equal to EH, and AK, KL, &c. equal to AE; and let the Parallelograms LO, K Φ , HX, MS, as likewise the Solids LP, KR, H Ω , MT, be completed: Then, because the right Lines LK, KA, AE, are equal; the Parallelograms LO, K Φ , AF, shall be $*$ also equal; as likewise the Pa- $*$ 1. 6. rallelograms K Ξ , KB, AG: And, moreover, $\dagger$ the $\dagger$ 24 of this. Parallelograms L Ψ , KP, AR, for they are opposite. For the same Reason, the Parallelograms EC, HX, MS, also, are equal to each other; as also the Parallelograms

lelograms HG , HI , IN ; and so are the Parallelograms DH , $M\Omega$, NT : Therefore three Planes of the Solid LP are equal to three Planes of the Solid KR , or AY , each to each; and the Planes opposite to these are equal to them: Therefore the three Solids LP , KR , AY , will be equal * to each other. For the same Reason, the three Solids ED , $H\Omega$, MT , are equal to each other; therefore the Base LF is the same Multiple of the Base AF , as the Solid LY is of the Solid AY . For the same Reason, the Base NF is the same Multiple of the Base HF , as the Solid NY is of the Solid ED ; and if the Base LF be equal to the Base NF , the Solid LY shall be equal to the Solid NY ; and if the Base LF exceeds the Base NF , the Solid LY shall exceed the Solid NY ; and, if it be less, less: Wherefore, because there are four Magnitudes, viz. the two Bases AF , FH , and the two Solids AY , ED , whose Equimultiples are taken, to wit, the Base LF , and the Solid LY ; and the Base NF , and the Solid NY : And since it is prov'd, if the Base LF exceeds the Base NF , then the Solid LY will exceed the Solid NY ; if equal, equal; and less, less: Therefore, as the Base AF is to the Base FH , so is * the Solid AY to the Solid ED . Wherefore, *if a solid Parallelopipedon be cut by a Plane, parallel to opposite Planes; then, as Base is to Base, so shall Solid be to Solid; which was to be demonstrated.*

PROPOSITION XXVI.

THEOREM.

At a right Line given, and at a Point given in it, to make a solid Angle equal to a solid Angle given.

LET AB be a right Line given, A a given Point in it, and D a given solid Angle contain'd under the plain Angles EDC , EDF , FDC ; it is requir'd to make a solid Angle at the given Point A , in the given right Line AB , equal to the given solid Angle D .

Assume any Point F in the right Line DF , from which let FG be drawn * perpendicular to the Plane passing

passing thro' ED, DC, meeting the said Plane in the Point G, and join DG; make † the Angles BAL, † 23. 1. BAK, at the given Point A, with the right Line AB, equal to the Angles EDC, EDG.

Lastly, Make AK equal to DG, and at the Point K erect † HK at right Angles to the Plane passing † 12 of *this*. thro' BAL; and make KH equal to GF; and join HA. I say, the solid Angle at A, which is contain'd under the three plain Angles BAL, BAH, HAL, is equal to the solid Angle at D, which is contain'd under the plain Angles EDC, EDF, FDC: For let the equal right Lines AB, DE, be taken; and join HB, KB, FE, GE: Then, because FG is perpendicular to the Plane passing through ED, DC, it shall be * perpendicular to all the right Lines touching it, that are in the said Plane. : Wherefore both the Angles FGD, FGE, are right Angles. For the same Reason, both the Angles HKA, HKB, are right Angles; and because the two Sides KA, AB, are equal to the two Sides GD, DE, each to each, and contain equal Angles, the Base BK shall be † equal to the † 4. 1. Base EG: But KH is also equal to GF, and they contain right Angles; therefore HB shall be † equal to FE. Again, because the two Sides AK, KH, are equal to the two Sides DG, GF, and they contain right Angles; the Base AH shall be equal to the Base DF: But AB is equal to DE; therefore the two Sides HA, AB, are equal to the two Sides FD, DE. But the Base HB is equal to the Base FE; and so the Angle BAH will be † equal to the Angle † 8. 1. EDF. For the same Reason, the Angle HAL is equal to the Angle FDC. For since, if AL be taken equal to DC; and KL, HL, GC, FC, be join'd; the whole Angle BAL is equal to the whole Angle EDC; and the Angle BAC, a Part of the one, is put equal to the Angle EDG, a Part of the other; the Angle KAL, remaining, will be equal to the Angle GDC remaining. And because the two Sides KA, AL, are equal to the two Sides GD, DC, and they contain equal Angles; the Base KL will be equal to the Base GC: But KH is equal to GF; wherefore the two Sides LK, KH, are equal to the two Sides CG, GF: But they contain right Angles; therefore the Base HL will be equal to the Base FC.

Again, because the two Sides HA , AL , are equal to the two Sides FD , DC ; and the Base HL is equal to the Base FC ; the Angle HAL will be equal to the Angle FDC : But the Angle BAL is equal to the Angle EDC ; *which was to be done.*

PROPOSITION XXVII.

PROBLEM.

Upon a right Line given, to describe a Parallelopipedon similar, and in like manner situate to a solid Parallelopipedon given.

LET AB be a right Line, and CD a given solid Parallelopipedon. It is requir'd to describe a solid Parallelopipedon upon the given right Line AB , similar and alike situate to the given solid Parallelopipedon CD .

Make a solid Angle at the given Point A , in the
 * 26 of this. right Line AB , which * is contain'd under the Angles BAH , HAK , KAB ; so that the Angle BAH be equal to the Angle ECF , the Angle BAK to the Angle ECG , and the Angle HAK to the Angle GCF ; and make, as EC is to CG , so BA † to AK ; and GC to CF , as KA to AH : Then, (by Equality of Proportion) as EC is to CF , so shall BA be to AH : Complete the Parallelogram BH , and the Solid AL ; then, because it is, as EC is to CG , so is BA to AK , *viz.* the Sides about the equal Angles ECG , BAK , proportional; the Parallelogram KB shall be similar to the Parallelogram GE . Also, for the same Reason, the Parallelogram KH shall be similar to the Parallelogram GF , and the Parallelogram HB to the Parallelogram FE : Therefore three Parallelograms of the Solid AL are similar to three Parallelograms of the Solid CD . But these three Parallelograms are † equal and similar to their three opposite ones; therefore the whole Solid AL will be similar to the whole Solid CD ; and so a solid Parallelopipedon AL is describ'd upon the given right Line AB , similar and alike situate to the given solid Parallelopipedon CD ; *which was to be done.*

† Cor. 24. of this.

PROPOSITION XXVIII.

THEOREM.

If a solid Parallelopipedon be cut by a Plane passing thro' the Diagonals of two opposite Planes, that Solid will be bisected by the Plane.

LET the solid Parallelopipedon AB be cut by the Plane CDEF, passing thro' the Diagonals CF, DE, of two opposite Planes. I say, the Solid AB is bisected by the Plane CDEF.

For, because the Triangle CGF is * equal to the * 34. 1. Triangle CBF, and the Triangle ADE to the Triangle DEH, and the Parallelogram CA to † the † 24 of this. Parallelogram BE, for it is opposite to it; and the Parallelogram GE to the Parallelogram CH; the Prism contain'd by the two Triangles CGF, ADE, and the three Parallelograms GE, AC, CE, is equal to the Prism contain'd under the two Triangles CBF, DEH, and the three Parallelograms CH, BE, CE; for they are contain'd under Planes equal in Number and Magnitude: Therefore the whole Solid AB is bisected by the Plane CDEF; *which was to be demonstrated.*

PROPOSITION XXIX.

THEOREM.

Solid Parallelopipedons, being constituted upon the same Base, and having the same Altitude, and whose insistent Lines are in the same right Lines, are equal to one another.

LET the solid Parallelopipedons CM, CN, be constituted upon the same Base AB, with the same Altitude, whose insistent Lines AF, AG, LM, LN, CD, CE, BH, BK, are in the same right Lines FN, DK. I say, the Solid CM is equal to the Solid CN.

For, because CH, CK, are both Parallelograms, CB shall be * equal to DH, or EK; wherefore DH * 34. 1.
is

is equal to EK . Let EH , which is common, be taken away; then the Remainder DE will be equal to the Remainder HK , and so the Triangle DEC is † equal to the Triangle HKB , and the Parallelogram DG equal to the Parallelogram HN . For the same Reason, the Triangle AFG is equal to the Triangle LMN , and the Parallelogram CF ‡ to the Parallelogram BM , and the Parallelogram CG to the Parallelogram BN , for they are opposite: Therefore the Prism contain'd under the two Triangles AFG , DEC , and the three Parallelograms AD , DG , GC , is equal to the Prism contain'd under the two Triangles LMN , HKB , and the three Parallelograms BM , NH , BN . Let the common Solid, whose Base is the Parallelogram AB , opposite to the Parallelogram $GEHM$, be added; then the whole solid Parallelopipedon CM is equal to the whole solid Parallelopipedon CN . Therefore, *solid Parallelopipedons, being constituted upon the same Base, and having the same Altitude, and whose insistent Lines are in the same right Lines, are equal to one another*; which was to be demonstrated.

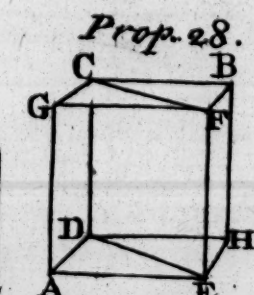
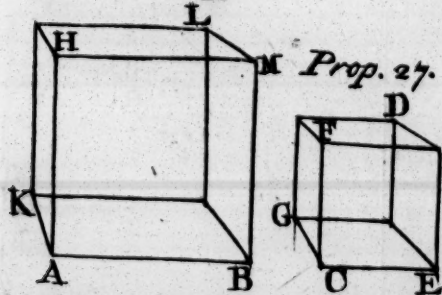
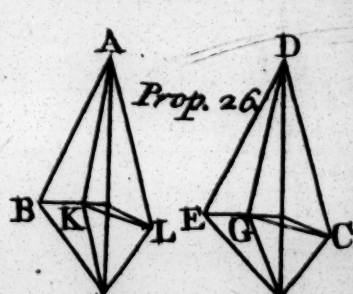
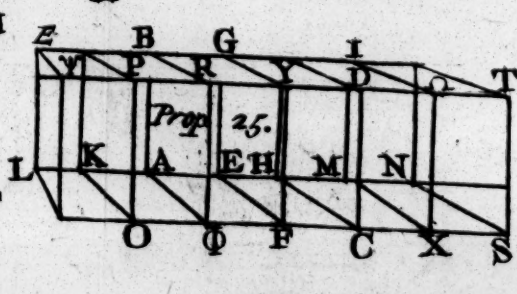
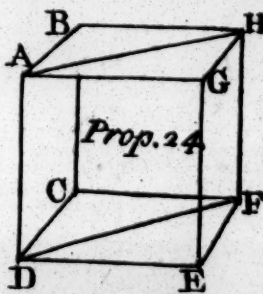
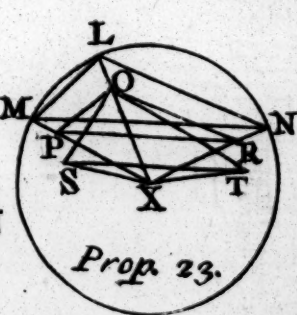
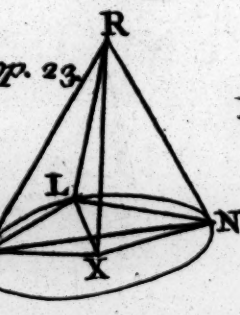
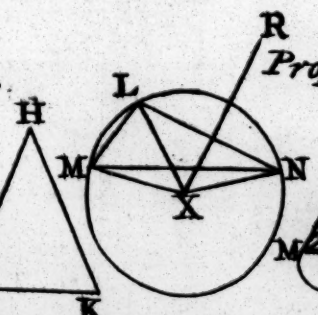
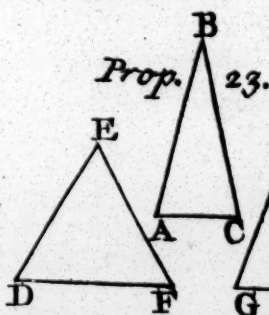
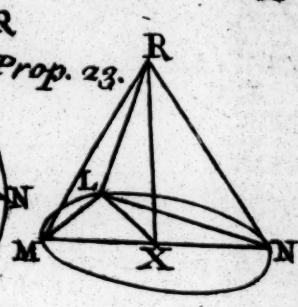
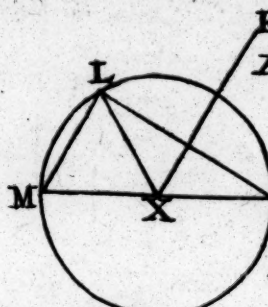
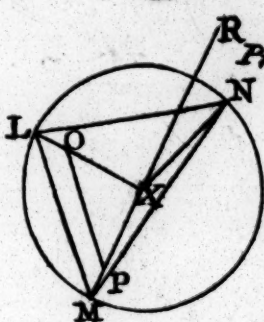
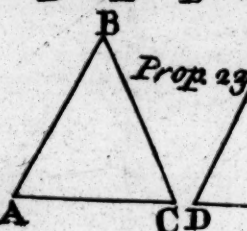
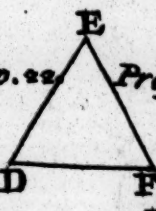
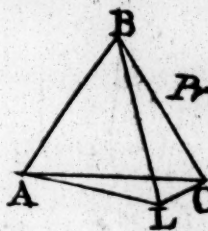
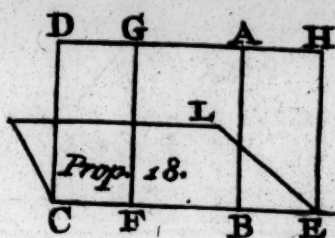
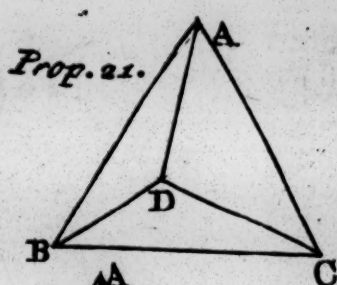
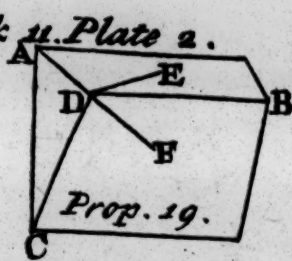
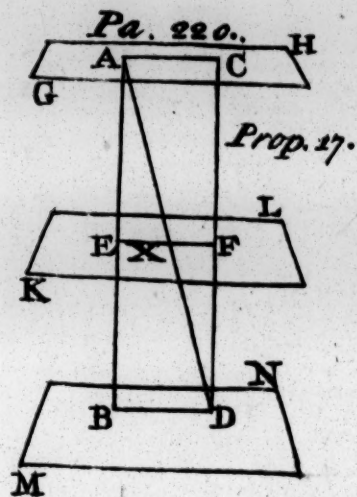
PROPOSITION XXX.

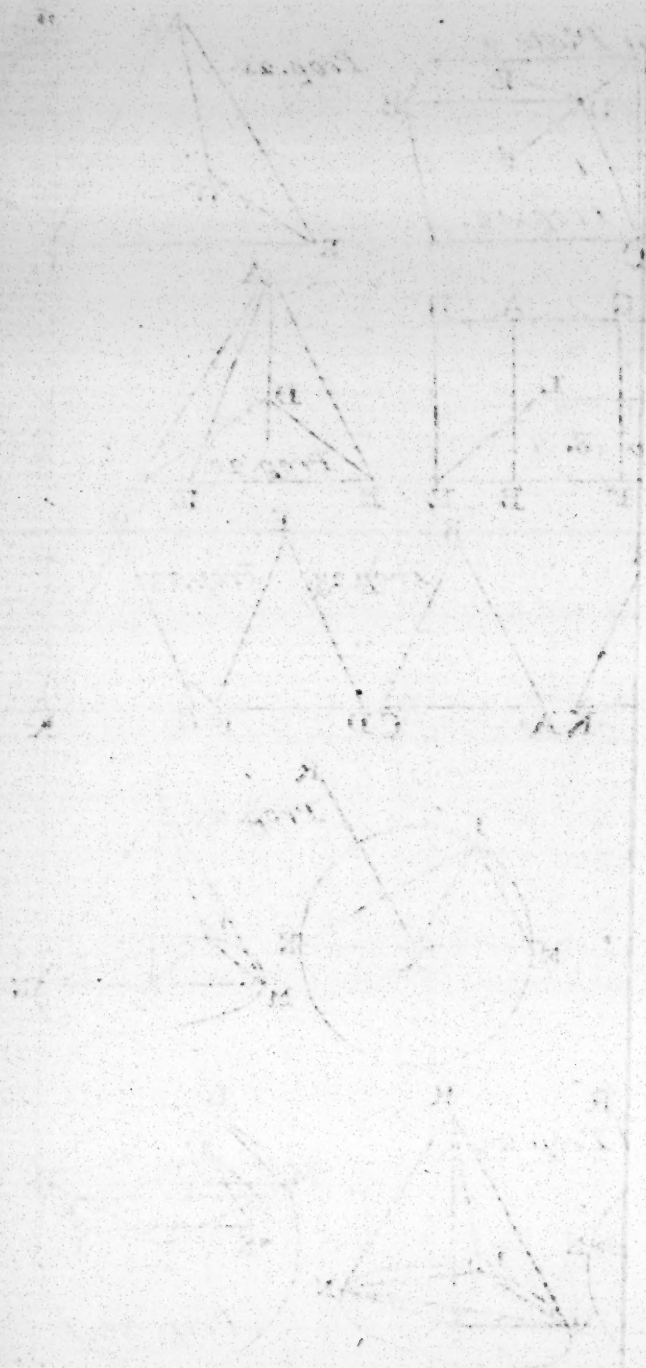
THEOREM.

Solid Parallelopipedons, being constituted upon the same Base, and having the same Altitude, whose insistent Lines are not placed in the same right Lines, are equal to one another.

LET there be solid Parallelopipedons CM , CN , having equal Altitudes, and standing on the same Base AB , and whose insistent Lines AF , AG , LM , LN , CD , CE , BH , BK , are not in the same right Lines. I say, the Solid CM is equal to the Solid CN .

For, let NK , DH , and GE , FM , be produced, meeting each other in the Points R , X ; let, also, FM , GE , be produced to the Points O , P , and join AX , LO , CP , BR . The Solid CM , whose Base is the Parallelogram $ACBL$, being opposite to the Parallelogram $FDHM$, is * equal to the Solid CO , whose





whose Base is the Parallelogram $ACBL$, being opposite to $XPRO$, for they stand upon the same Base $ACBL$; and the insistent Lines AF , AX , LM , LO , CD , CP , BH , BR , are in the same right Lines FO , DR : But the Solid CO , whose Base is the Parallelogram $ACBL$, being opposite to $XPRO$, is * equal to the Solid CN , whose Base is the Parallelogram $ACBL$, being opposite to $GEKN$; for they stand upon the same Base $ACBL$, and their insistent Lines AG , AX , CE , CP , LN , LO , BK , BR , are in the same right Lines GP , NR : Wherefore the Solid CM shall be equal to the Solid CN . Therefore, *solid Parallelopipedons, being constituted upon the same Base, and having the same Altitude, whose insistent Lines are not placed in the same right Lines, are equal to one another*; which was to be demonstrated.

PROPOSITION XXXI.

THEOREM.

Solid Parallelopipedons, being constituted upon equal Bases, and having the same Altitude, are equal to one another.

LET AE , CF , be solid Parallelopipedons constituted upon the equal Bases AB , CD , and having the same Altitude. I say, the Solid AE is equal to the Solid CF .

First, Let HK , BE , AG , LM , OP , DF , $CΞ$, RS , be at right Angles to the Bases AB , CD ; let the Angle ALB not be equal to the Angle CRD , and produce CR to T , so that RT be equal to AL ; then make the Angle TRY , at the Point R , in the right Line RT , equal * to the Angle ALB ; make RY equal to LB ; draw XY , thro' the Point Y , parallel to RT , and complete the Parallelogram RX , and the Solid $ΨY$. Therefore, because the two Sides TR , RY , are equal to the two Sides AL , LB , and they contain equal Angles; the Parallelogram RX shall be equal and similar to the Parallelogram HL . And, again, because AL is equal to RT , and LM to RS , and they contain equal Angles, the Parallelogram

lelogram $R\psi$ shall be equal and similar to the Parallelogram AM . For the same Reason, the Parallelogram LE is equal and similar to the Parallelogram SY ; therefore three Parallelograms of the Solid AE are equal and similar to three Parallelograms of the Solid ψY ; and so the three opposite ones of one Solid are † also equal and similar to the three opposite ones of the other: Therefore the whole solid Parallelopipedon AE is equal to the whole solid Parallelopipedon ψY . Let DR , XY , be produced, and meet each other in the Point Ω , and let TQ be drawn thro' T parallel to $D\Omega$, and produce TQ , OD , till they meet in V , and complete the Solid $\Omega\psi RI$: Then the Solid $\psi\Omega$, whose Base is the Parallelogram $R\psi$, and $\Omega\Gamma$ is that opposite to it, is ‡ equal to the Solid ψY , whose Base is the Parallelogram $R\psi$, and $Y\Phi$ is that opposite to it, for they stand upon the same Base $R\psi$, have the same Altitude, and their insistent Lines $R\Omega$, RY , TQ , TX , SZ , SN , $\psi\Gamma$, $\psi\Phi$, are in the same right Lines ΩX , $Z\Phi$: But the Solid ψY is equal to the Solid AE ; and so AE is equal to the Solid $\psi\Omega$. Again, because the Parallelogram $RYXT$ is equal to the Parallelogram ΩT , for it stands on the same Base RT , and between the same Parallels RT , ΩX ; and the Parallelogram $RYXT$ is equal to the Parallelogram CD , because it is also equal to AB ; the Parallelogram ΩT is equal to the Parallelogram CD , and DT is some other Parallelogram: Therefore, as the Base CD is to the Base DT , so is ΩT to TD . And because the solid Parallelopipedon CI is cut by the Plane RF , being parallel to two opposite Planes; it shall be *, as the Base CD is to the Base DT , so is the Solid CF to the Solid RI . For the same Reason, because the solid Parallelopipedon ΩI is cut by the Plane $R\psi$ parallel to two opposite Planes; as the Base ΩT is to the Base DT , so shall * the Solid $\Omega\psi$ be to the Solid RI : But as the Base CD is to the Base DT , so is the Base ΩT to TD : Therefore, as the Solid CF is to the Solid RI , so is the Solid $\Omega\psi$ to the Solid RI . And since each of the Solids CF , $\Omega\psi$, has the same Proportion to the Solid RI , the Solid CF is equal to the Solid $\Omega\psi$: But the Solid $\Omega\psi$ has been prov'd equal to the Solid AE ; there-

fore the Solid A E shall be † equal to the Solid C F. † 9. 5.
 But, now let the insistent Lines A G, H K, B E, L M, C N, O P, D F, R S, not be at right Angles to the Bases A B, C D. I say, again, that the Solid A E is equal to the Solid C F. Let there be drawn from the Points K, E, G, M, P, F, N, S, to the Plane wherein are the Bases A B, C D, the Perpendiculars K Ξ , E T, G Y, M Φ , S I, F Ψ , N Ω , P X, meeting the Plane in the Points Ξ , T, Y, Φ , I, Ψ , Ω , X; and join Ξ T, Y Φ , Ξ Y, T Φ , X Ψ , X Ω , Ω I, Ψ I; then the Solid K Φ is equal to the Solid P I; for they stand on equal Bases K M, P S, have the same Altitude, and the insistent Lines are at right Angles to the Bases: But the Solid K Φ is equal to the Solid A E, and the Solid P I to † the Solid C F, since they stand upon the † 29 of this same Base, have the same Altitude, and their insistent Lines are in the same right Lines: Therefore the Solid A E shall be equal to the Solid C F. Wherefore, *solid Parallelopipedons, being constituted upon equal Bases, and having the same Altitude, are equal to one another*; which was to be demonstrated.

PROPOSITION XXXII.

THEOREM.

Solid Parallelopipedons, that have the same Altitude, are to each other as their Bases.

LET A B, C D, be solid Parallelopipedons, that have the same Altitude. I say, they are to one another, as their Bases; that is, as the Base A E is to the Base C F, so is the Solid A B to the Solid C D.

For, apply a Parallelogram F H to the right Line F G, equal to the Parallelogram A E; and complete the solid Parallelopipedon G K upon the Base F H, having the same Altitude as C D has: Then the Solid A B is * equal to the Solid G K, for they stand * 31 of this upon equal Bases A E, F H, and have the same Altitude; and so, because the solid Parallelopipedon C K is cut by the Plane D G, parallel to two opposite Planes, it shall be †, as the Base H F is to the Base † 25 of this F C, so is the Solid H D to the Solid D C: But the
 Base

Base FH is equal to the Base AE, and the Solid AB to the Solid FK. Therefore, as the Base AE is to the Base CF, so is the Solid AB to the Solid CD. Wherefore, *solid Parallelopipedons, that have the same Altitude, are to each other as their Bases*; which was to be demonstrated.

PROPOSITION XXXIII.

THEOREM.

Similar solid Parallelopipedons are to one another in the triplicate Proportion of their homologous Sides.

LET AB, CD, be solid Parallelopipedons, and let the Side AE be homologous to the Side CF. I say, the Solid AB, to the Solid CD, hath a Proportion triplicate of that which the Side AE has to the Side CF.

For, produce AE, GE, HE, to EK, EL, EM; and make EK equal to CF, and EL to FN, and EM to FR; and let the Parallelogram KL, and likewise the Solid KO, be completed: Then, because the two Sides KE, EL, are equal to the two Sides CF, FN; and the Angle KEL equal to the Angle CFN: Since the Angle AEG is also equal to the Angle CFN, because of the Similarity of the Solids AB, CD, the Parallelogram KL shall be similar and equal to the Parallelogram CN. For the same Reason, the Parallelogram KM is equal and similar to the Parallelogram CR, and the Parallelogram OE to DF; therefore three Parallelograms of the Solid KO are equal and similar to three Parallelograms of the Solid CD: But those three Parallelograms are * equal and similar to the three opposite Parallelograms; therefore the whole Solid KO is equal and similar to the whole Solid CD. Let the Parallelogram GK be completed, as also the Solids EX, LP, upon the Bases GK, KL, having the same Altitude as AB: And since, because of the Similarity of the Solids AB and CD, it is, as AE is to CF, so is EG to FN; and so EH to FR; and FC is equal to EK, and FN to EL, and FR to EM; it shall be, as AE is to EK,

* 24 of this.

EK, so is † the Parallelogram AG to the Parallelo- † 1. 6.
 gram GK; but as GE is to EL, so is GK to KL;
 and as HE is † to EM, so is PE to KM: There-
 fore, as the Parallelogram AG is to the Parallelogram
 GK, so is GK to KL, and PE to KM. But as
 AG is to GK, so is † the Solid AB to the Solid EX; † 32 of this,
 and as GK is to KL, so is the Solid EX to the Solid
 PL; and as PE is to KM, so is the Solid PL to
 the Solid KO: Therefore, as the Solid AB is to the
 Solid EX, so is * EX to PL, and PL to KO: But * 11. 5.
 if four Magnitudes be continually proportional, the
 first to the fourth hath † a triplicate Proportion of † Def. 11. 5.
 that which it has to the second. Therefore, also, the
 Solid AB, to the Solid KO, hath a triplicate Propor-
 tion of that which AB has to EX: But as AB is to
 EX, so is the Parallelogram AG to the Parallelogram
 GK; and so is the right Line AE to the right
 Line EK: Wherefore the Solid AB, to the Solid
 KO, hath a Proportion triplicate of that which AE
 has to EK. But the Solid KO is equal to the Solid
 CD, and the right Line EK equal to the right Line
 CF; therefore the Solid AB, to the Solid CD, has
 a Proportion triplicate of that which the homologous
 Side AE has to the homologous Side CF; *which was
 to be demonstrated.*

Coroll. From hence it is manifest, if four right Lines
 be proportional, as the first is to the fourth, so is
 a solid Parallelopipedon describ'd upon the first,
 to a similar solid Parallelopipedon, alike situate, de-
 scrib'd upon the second; because the first to the
 fourth has a Proportion triplicate of that which it
 has to the second.

PROPOSITION XXXIV.

THEOREM.

The Bases and Altitudes of equal solid Parallelopipedons, are reciprocally proportional; and those solid Parallelopipedons, whose Bases and Altitudes are reciprocally proportional, are equal.

LET AB, CD, be equal solid Parallelopipedons: I say, their Bases and Altitudes are reciprocally proportional; that is, as the Base EH is to the Base NP, so is the Altitude of the Solid CD to the Altitude of the Solid AB.

First, Let the insistent Lines AG, EF, LB, HK, CM, NX, OD, PR, be at right Angles to their Bases: I say, as the Base EH is to the Base NP, so is CM to AG. For if the Base EH be equal to the Base NP, and the Solid AB is equal to the Solid CD; the Altitude CM shall also be equal to the Altitude AG: For if, when the Bases EH, NP, are equal, the Altitudes AG, CM, are not so; then the Solid AB will not be equal to the Solid CD, but it is put equal to it: Therefore the Altitude CM is not unequal to the Altitude AG, and so they are necessarily equal to one another; and, consequently, as the Base EH is to the Base NP, so shall CM be to AG. But now let the Base EH be unequal to the Base NP, and let EH be the greater; then, since the Solid AB is equal to the Solid CD, CM is greater than AG; for, otherwise, it would follow, that the Solids AB, CD, are not equal, which are put such: Therefore, make CT equal to AG, and complete the solid Parallelopipedon VC upon the Base NP, having the Altitude CT: Then, because the Solid AB is equal to the Solid CD, and VC is some other Solid; and since equal Magnitudes have * the same Proportion to the same Magnitude; it shall be, as the Solid AB is to the Solid CV, so is the Solid CD to the Solid CV; But as the Solid AB is to the Solid CV, † 32 of this. so is ‡ the Base EH to the Base NP; for AB, CV, are Solids having equal Altitudes: And as the Solid † 25 of this. CD is to the Solid CV, so is ‡ the Base MP to the Base

Base PT , and so is MC to CT : Therefore, as the Base EH is to the Base NP , so is MC to CT . But CT is equal to AG ; wherefore, as the Base EH is to the Base NP , so is MC to AG : Therefore the Bases and Altitudes of the equal solid Parallelepipedons AB , CD , are reciprocally proportional.

Now, let the Bases and Altitudes of the solid Parallelepipedons AB , CD , be reciprocally proportional; that is, let the Base EH be to the Base NP , as the Altitude of the Solid CD is to the Altitude of the Solid AB . I say, the Solid AB is equal to the Solid CD .

For, let again the insistent Lines be at right Angles to the Bases; then, if the Base EH be equal to the Base NP , and EH is to NP as the Altitude of the Solid CD is to the Altitude of the Solid AB ; the Altitude of the Solid CD shall be equal to the Altitude of the Solid AB . But solid Parallelepipedons, that stand upon equal Bases, and have the same Altitude, are * equal to each other; therefore the Solid * 31 of this AB is equal to the Solid CD .

But now let the Base EH not be equal to the Base NP , and let EH be the greater; then the Altitude of the Solid CD is greater than the Altitude of the Solid AB ; that is, CM is greater than AG : Again, put CT equal to AG , and complete the Solid CV , as before; and then, because the Base EH is to the Base NP , as MC is to AG , and AG is equal to CT ; it shall be, as the Base EH is to the Base NP , so is MC to CT : But as the Base EH is to the Base NP , so is the Solid AB to the Solid VC , for the Solids AB , CV , have equal Altitudes; and as MC is to CT , so is the Base MP to the Base PT , and so the Solid CD to the Solid CV : Therefore, as the Solid AB is to the Solid CV , so is the Solid CD to the Solid CV : But since each of the Solids AB , CD , has the same Proportion to CV , the Solid AB shall be equal to the Solid CD ; *which was to be demonstrated.*

Now, let the insistent Lines FE , BL , GA , KH , XN , DO , MC , RP , not be at right Angles to the Bases; and from the Points F , G , B , K , X , M , D , R , let there be drawn Perpendiculars to the Planes of the Bases EH , NP , meeting the same in the

Q
Points

Points S, T, Y, V, Q, Z, Ω , Φ , and complete the Solids FV, X Ω . Then, I say, if the Solids AB, CD, be equal, their Bases and Altitudes are reciprocally proportional, *viz.* as the Base EH is to the Base NP, so is the Altitude of the Solid CD to the Altitude of the Solid AB.

For, because the Solid AB is equal to the Solid CD, and the Solid AB is * equal to the Solid BT, for they stand upon the same Base, have the same Altitude, and their insistent Lines are not in the same right Lines, and the Solid DC is * equal to the Solid DZ, since they stand upon the same Base XR, have the same Altitude, and their insistent Lines are not in the same right Lines; the Solid BT shall be equal to the Solid DZ. But the Bases and Altitudes of those equal Solids whose Altitudes are at right Angles to their Bases, are † reciprocally proportional; therefore, as the Base FK is to the Base XR, so is the Altitude of the Solid DZ, to the Altitude of the Solid BT. But the Base FK is equal to the Base EH, and the Base XR to the Base NP; wherefore, as the Base EH is to the Base NP, so is the Altitude of the Solid DZ to the Altitude of the Solid BT. But the Solids DZ, DC, have the same Altitude, and so have the Solids BT, BA; therefore the Base EH is to the Base NP, as the Altitude of the Solid DC is to the Altitude of the Solid AB; and so the Bases and Altitudes of equal Solids are reciprocally proportional.

† From
what has
been before
proved.

Again, let the Bases and Altitudes of the solid Parallelepipedons AB, CD, be reciprocally proportional, *viz.* as the Base EH is to the Base NP, so let the Altitude of the Solid CD be to the Altitude of the Solid AB: I say, the Solid AB is equal to the Solid CD.

For, the same Construction remaining, because the Base EH is to the Base NP, as the Altitude of the Solid CD is to the Altitude of the Solid AB; and since the Base EH is equal to the Base FK, and NP to XR; it shall be, as the Base FK is to the Base XR, so is the Altitude of the Solid CD to the Altitude of the Solid AB. But the Altitudes of the Solids AB, BT, are the same; as also of the Solids CD, DZ; therefore the Base FK is to the Base XR, as the Altitude

Altitude of the Solid DZ is to the Altitude of the Solid BT ; wherefore the Bases and Altitudes of the solid Parallelepipedons BT , DZ , are reciprocally proportional; but those solid Parallelepipedons whose Altitudes are at right Angles to their Bases, and the Bases and Altitudes are reciprocally proportional, are equal to each other. But the Solid BT is equal to the Solid BA , for they stand upon the same Base FK , have the same Altitude, and their insistent Lines are not in the same right Lines; and the Solid DZ is also equal to the Solid DC , since they stand upon the same Base XR , have the same Altitude, and their insistent Lines are not in the same right Lines: Therefore the Solid AB is equal to the Solid CD ; which was to be demonstrated.

PROPOSITION XXXV.

THEOREM.

If there be two plain Angles equal, and from the Vertices of those Angles two right Lines be elevated above the Planes, in which the Angles are, containing equal Angles with the Lines first given, each to its correspondent one; and if in those elevated Lines any Points be taken, from which Lines be drawn perpendicular to the Planes in which the Angles first given are, and right Lines be drawn to the Angles first given from the Points made by the Perpendiculars in the Planes; those right Lines will contain equal Angles with the elevated Lines.

LET BAC , EDF , be two equal right-lin'd Angles; and from A , D , the Vertices of those Angles, let two right Lines, AG , DM , be elevated above the Planes of the said Angles, making equal Angles with the Lines first given, each to its correspondent one, viz. the Angle MDE equal to the Angle GAB ; and the Angle MDF to the Angle GAC ; and take any Points G and M in the right Lines AG , DM , from which let GL , MN , be drawn perpendicular to the Planes passing thro' BAC , EDF , meeting the same in the Points L , N ; and

Q 2

join

join LA, ND. I say, the Angle GAL is equal to the Angle MDN.

- Make AH equal to DM, and thro' H let HK be drawn parallel to GL; but GL is perpendicular to the Plane passing thro' BAC; therefore HK shall be * also perpendicular to the Plane passing thro' BAC: Draw, from the Points K, N, to the right Lines AB, AC, DE, DF, the Perpendiculars KB, KC, NE, NF; and join HC, CB, MF, FE: Then, because the Square of HA is † equal to the Squares of HK, KA; and the Squares of KC, CA, are † equal to the Square of KA; the Square of HA shall be equal to the Squares of HK, KC, CA: But the Square of HC is equal to the Squares of HK, KC; therefore the Square of HA will be equal to the Squares of HC and CA; and so the Angle HCA is † a right Angle. For the same Reason, the Angle DFM is also a right Angle; therefore the Angle ACH is equal to DFM: But the Angle HAC is also equal to the Angle MDF; therefore the two Triangles MDF, HAC, have two Angles of the one equal to two Angles of the other, each to each, and one Side of the one equal to one Side of the other, viz. that which is subtended by one of the equal Angles, that is, the Side HA equal to DM; and so the other Sides of the one shall be * equal to the other Sides of the other, each to each: Wherefore AC is equal to DF. In like manner we demonstrate, that AB is equal to DE: For, let HB, ME, be join'd; then, because the Square of AH is equal to the Squares of AK and KH; and the Squares of AB, BK, are equal to the Square of AK; the Squares of AB, BK, KH, will be equal to the Square of AH. But the Square of BH is equal to the Squares of BK, KH; for the Angle HKB is a right Angle, because HK is perpendicular to the Plane passing through BAC; therefore the Square of AH is equal to the Squares of AB, BH: Wherefore the Angle ABH is † a right Angle. For the same Reason, the Angle DEM is also a right Angle; and the Angle BAH is equal to the Angle EDM, for so it is put; and AH is equal to DM; therefore AB is † also equal to DE: And so, since AC is equal to DF; and AB to DE; the two Sides CA, AB, shall be equal to the

the two Sides FD , DE : But the Angle BAC is equal to the Angle FDE ; therefore the Base BC is * equal to the Base EF , the Triangle to the Tri- * 4. 1.
angle, and the other Angles to the other Angles: Wherefore the Angle ACB is equal to the Angle DFE . But the right Angle ACK is equal to the right Angle DFN ; and therefore the remaining Angle BCK is equal to the remaining Angle EFN . For the same Reason, the Angle CBK is equal to the Angle FEN ; and so, because BCK , EFN , are two Triangles, having two Angles equal to two Angles, each to each, and one Side equal to one Side, which is at the equal Angles, *viz.* BC equal to EF ; therefore they shall have the other Sides equal to the other Sides: Therefore CK is equal to FN . But AC is equal to DF ; therefore the two Sides AC , CK , are equal to the two Sides DF , FN , and they contain right Angles; consequently, the Base AK is equal to the Base DN . And since AH is equal to DM , the Square of AH shall be equal to the Square of DM : But the Squares of AK , KH , are equal to the Square of AH , for the Angle AKH is a right Angle; and the Squares DN , NM , are equal to the Square of DM , since the Angle DNM is a right Angle: Therefore the Squares of AK , KH , are equal to the Squares of DN , NM ; of which the Square of AK is equal to the Square of DN : Wherefore the Square of KH remaining, is equal to the remaining Square of NM ; and so the right Line HK is equal to MN . And since the two Sides HA , AK , are equal to the two Sides MD , DN , each to each, and the Base HK has been prov'd equal to the Base NM , the Angle HAK shall be † equal to the Angle † 8. 1.
 MDN ; which was to be demonstrated.

Coroll. From hence it is manifest, that if there be two right-lin'd plain Angles equal, from whose Points equal right Lines be elevated on the Planes of the Angles, containing equal Angles with the Lines first given, each to each; Perpendiculars drawn from the extreme Points of those elevated Lines to the Planes of the Angles first given, are equal to one another.

PROPOSITION XXXVI.

THEOREM.

If three right Lines be proportional, the solid Parallelepipedon made of them is equal to the solid Parallelepipedon made of the middle Line, if it be an equilateral one, and equiangular to the aforesaid Parallelepipedon.

LET three right Lines A, B, C, be proportional; viz. let A be to B, as B is to C. I say, the Solid made of A, B, C, is equal to the equilateral Solid made of B, equiangular to that made on A, B, C.

Let E be a solid Angle contain'd under the three plain Angles DEG, GEF, FED; and make DE, GE, EF, each equal to B, and complete the solid Parallelepipedon EK: Again, put LM equal to A, and at the Point L, at the right Line LM, make * a solid Angle contain'd under the plain Angles NLX, XLM, MLN, equal to the solid Angle E; and make LN equal to B, and LX to C: Then, because A is to B as B is to C; and A is equal to LM; and B to LN, EF, EG, or ED; and C to LX; it shall be, as LM is to EF, so is GE to LX: And so the Sides about the equal Angles MLX, GEF, are reciprocally proportional. Wherefore the Parallelogram MX is † equal to the Parallelogram GF. And since the two plain Angles GEF, XLM, are equal, and the right Lines LN, ED, being equal, are erected at the angular Points containing equal Angles with the Lines first given, each to each; the Perpendiculars drawn ‡ from the Points N D, to the Planes drawn thro' XLM, GEF, are equal one to another: Therefore the Solids LH, EK, have the same Altitude. But solid Parallelepipedons, that have equal Bases, and the same Altitude, are * equal to each other; therefore the Solid HL is equal to the Solid EK. But the Solid HL is that made of three right Angles A, B, C; and the Solid EK that made of the right Line B: Therefore, *if three right Lines be proportional, the solid Parallelepipedon made of them is equal to the solid Parallelepipedon made of the middle Line, if it be an equi-*

* 26 of this.

† 14. 6.

‡ Cor. 35. of this.

* 31 of this.

equilateral one, and equiangular to the aforesaid Parallelepipedon ; which was to be demonstrated.

PROPOSITION XXXVII.

THEOREM.

If four right Lines be proportional, the solid Parallelepipedons similar, and in like manner described from them, shall be proportional. And if the solid Parallelepipedons, being similar, and alike described, be proportional, then the right Lines they are described from shall be proportional.

LET the four right Lines AB, CD, EF, GH, be proportional, viz. let AB be to CD, as EF is to GH ; and let the similar and alike situate Parallelepipedons KA, LC, ME, NG, be describ'd from them. I say, KA is to LC, as ME is to NG.

For, because the solid Parallelepipedon KA is similar to LC, therefore KA to LC shall have * a Pro-^{33 of this.} portion triplicate of that which AB has to CD. For the same Reason, the Solid ME to NG will have a triplicate Proportion of that which EF has to GH. But AB is to CD, as EF is to GH ; therefore AK is to LC, as ME is to NG : And if the Solid AK be to the Solid LC, as the Solid ME is to the Solid NG ; I say, as the right Line AB is to the right Line CD, so is the right Line EF to the right Line GH : For, because AK to LC has * a Proportion triplicate of^{33 of this.} that which AB has to CD ; and ME to NG has a Proportion triplicate of that which EF has to GH ; and since AK is to LC, as ME is to NG ; it shall be, as AB is to CD, so is EF to GH. Therefore, *if four right Lines be proportional, the solid Parallelepipedons similar, and in like manner described from them, shall be proportional. And if the solid Parallelepipedons, being similar and alike described, be proportional, then the right Lines they are described from shall be proportional ; which was to be demonstrated.*

PROPOSITION XXXVIII.

THEOREM.

If a Plane be perpendicular to a Plane, and a Line be drawn from a Point in one of the Planes perpendicular to the other Plane; that Perpendicular shall fall in the common Section of the Planes.

LET the Plane CD be perpendicular to the Plane AB, let their common Section be AD, and let some Point E be taken in the Plane CD. I say, a Perpendicular, drawn from the Point E to the Plane AB, falls on AD.

For, if it does not, let it fall without the same, as EF, meeting the Plane AB in the Point F; and from the Point F let FG be drawn in the Plane AB perpendicular to AD; this shall be * perpendicular to the Plane CD; and join EG: Then, because FG is perpendicular to the Plane CD, and the right Line EG, in the Plane of CD, touches it; the Angle FGE shall be † a right Angle. But EF is also at right Angles to the Plane AB; therefore the Angle EFG is a right Angle: And so two Angles of the Triangle EFG are equal to two right Angles, which is ‡ absurd. Wherefore a right Line, drawn from the Point E perpendicular to the Plane AB, does not fall without the right Line AD; and so it must necessarily fall on it. Therefore, *if a Plane be perpendicular to a Plane, and a Line be drawn from a Point in one of the Planes perpendicular to the other Plane; that Perpendicular shall fall in the common Section of the Planes; which was to be demonstrated.*

* Def. 4. of
this.

† Def. 3. of
this.

‡ 17. 1.

PROPOSITION XXXIX.

THEOREM.

If the Sides of the opposite Planes of a solid Parallelepipedon be divided into two equal Parts, and Planes be drawn thro' their Sections; the common Section of the Planes, and the Diameter of the solid Parallelepipedon, shall divide each other into two equal Parts.

LET the Sides of CF, AH, the opposite Planes of the solid Parallelepipedon AF, be cut in half in the Points K, L, M, N, X, O, P, R; and let the Planes KN, XR, be drawn through the Sections: Also, let YS be the common Section of the Planes, and DG the Diameter of the solid Parallelepipedon. I say, YS, DG, bisect each other; that is, YT is equal to TS, and DT to TG.

For, join DY, YE, BS, SG. Then, because DX is parallel to OE, the alternate Angles DXY, YOE, are * equal to one another. And because DX is equal * 29. 1. to OE, and YX to YO, and they contain equal Angles, the Base DY shall be † equal to the Base YE, † 4. 1. and the Triangle DXY to the Triangle YOE, and the other Angles equal to the other Angles: Therefore the Angle XYD is equal to the Angle OYE; and so DYE is † a right Line. For the same Reason, BSG is also a right Line, and BS is equal to SG; then, because CA is equal and parallel to DB, as also to EG, DB shall be equal and parallel to EG; and the right Lines DE, GB, join them: Therefore DE is * parallel to BG, and D, Y, G, S, are Points * 33. 1. taken in each of them; and DG, YS, are join'd: Therefore DG, YS, are † in one Plane. And since DE † 7 of this. is parallel to BG, the Angle EDT shall be * equal to * 29. 1. the Angle BGT, for they are alternate: But the Angle DTY is † equal to the Angle GTS; therefore † 15. 1. DTY, GTS, are two Triangles having two Angles of the one equal to two Angles of the other, as likewise one Side of the one equal to one Side of the other, viz. the Side DY equal to the Side GS, for they are Halves of DE, BG; therefore they shall

have the other Sides of one equal to the other Sides of the other ; and so DT is equal to TG , and YT to TS . Wherefore, if the Sides of the opposite Planes of a solid Parallelepipedon be divided into two equal Parts, and Planes be drawn thro' their Sections; the common Section of the Planes, and the Diameter of the solid Parallelepipedon, shall divide each other into two equal Parts ; which was to be demonstrated.

PROPOSITION XL.

THEOREM.

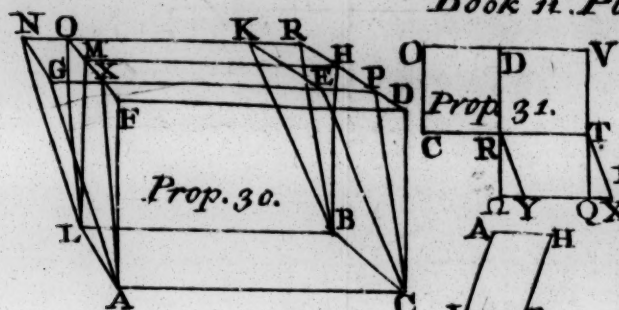
If of two triangular Prisms, one standing on a Base which is a Parallelogram, and the other on a Triangle, if their Altitudes from these Bases are equal, and the Parallelogram double to the Triangle; then those Prisms are equal to each other.

LET $ABCDEF$, $GHKLMN$, be two Prisms of equal Altitude, the Base of one of which is the Parallelogram AF ; and that of the other the Triangle GHK ; and let the Parallelogram AF be double to the Triangle GHK . I say, the Prism $ABCDEF$ is equal to the Prism $GHKLMN$.

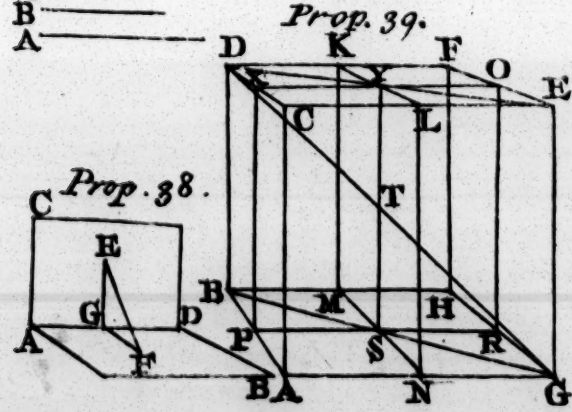
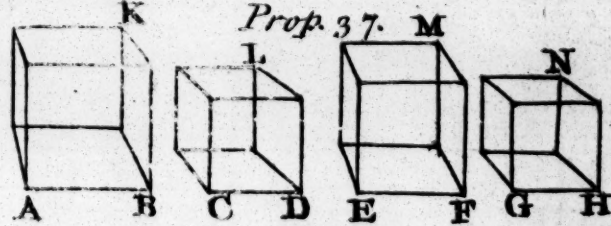
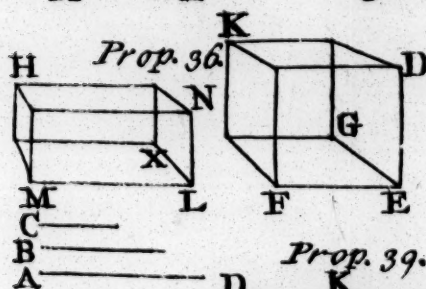
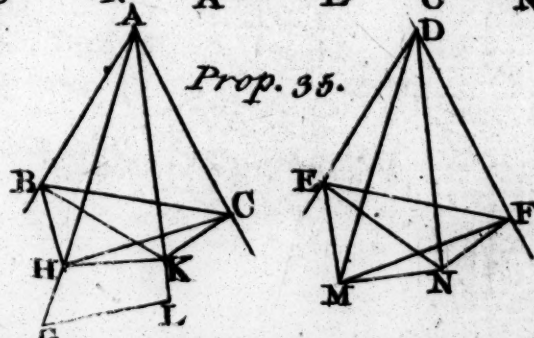
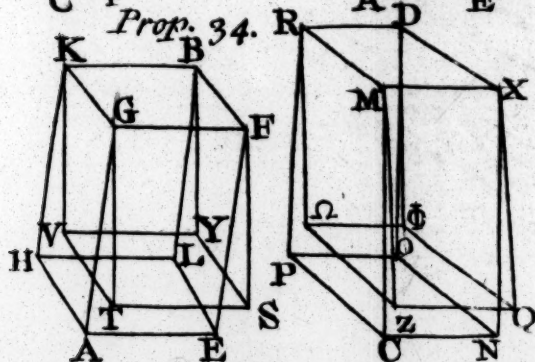
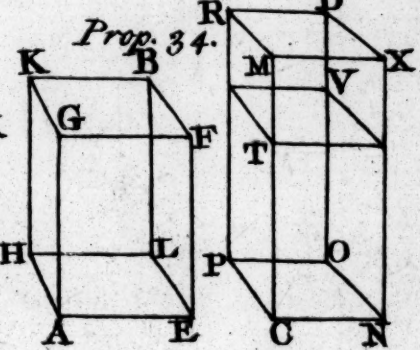
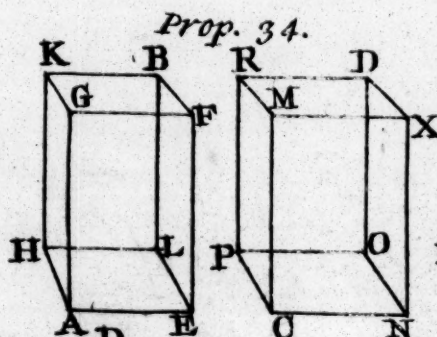
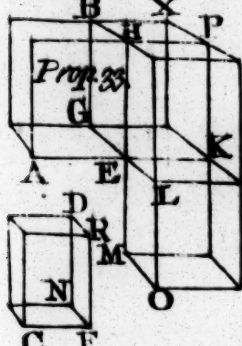
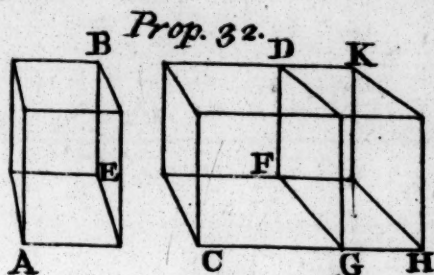
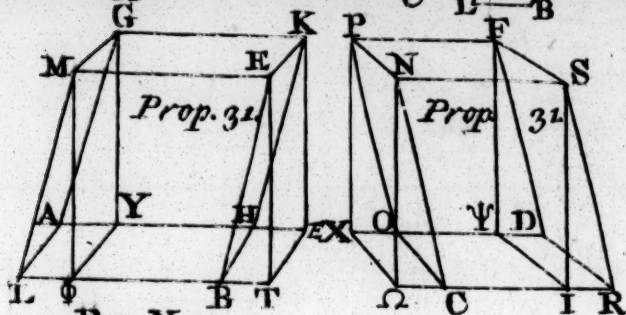
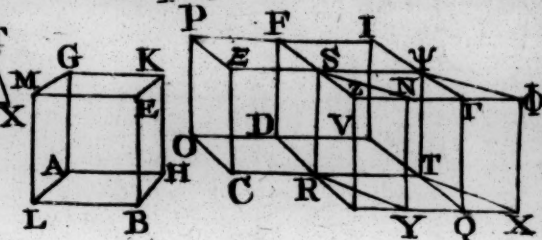
For, complete the Solids AX , GO . Then, because the Parallelogram AF is double to the Triangle GHK ; and since the Parallelogram HK is * double to the Triangle GHK , the Parallelogram AF shall be equal to the Parallelogram HK . But solid Parallelepipeds, that stand upon equal Bases, and have the same Altitude, are † equal to one another ; therefore the Solid AX is equal to the Solid GO . But the Prism $ABCDEF$ is half the Solid AX , and the Prism $GHKLMN$ is ‡ half the Solid GO ; therefore the Prism $ABCDEF$ is equal to the Prism $GHKLMN$. Wherefore, if there be two Prisms having equal Altitudes, the Base of one of which is a Parallelogram, and that of the other a Triangle, and if the Parallelogram be double to the Triangle, the said Prisms shall be equal to each other.

The END of the ELEVENTH BOOK.

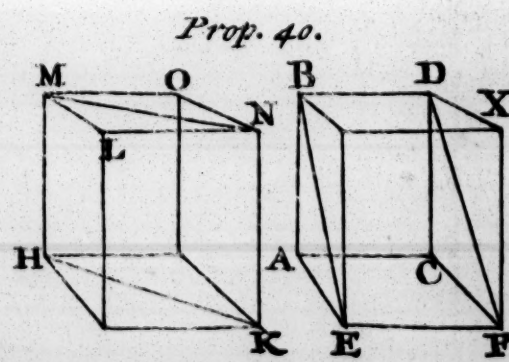
EUCLID'S



Prop. 31.



Prop. 39.





E U C L I D's

ELEMENTS.

B O O K XII.

PROPOSITION I.

T H E O R E M.

Similar Polygons, inscribed in Circles, are to one another as the Squares of the Diameters of the Circles.

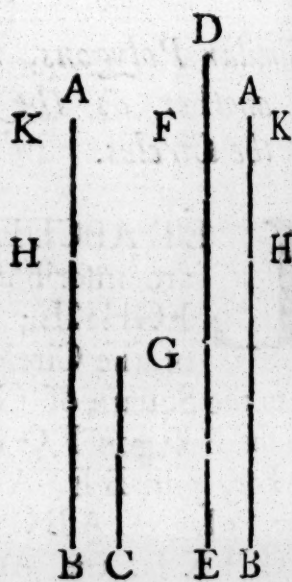
LET ABCDE, FGHL, be Circles, wherein are inscrib'd the similar Polygons ABCDE, FGHL; and let BM, GN, be Diameters of the Circles. I say, as the Square of BM is to the Square of GN, so is the Polygon ABCDE to the Polygon FGHL.

For, join BE, AM, GL, FN. Then, because the Polygon ABCDE is similar to the Polygon FGHL, the Angle BAE is equal to the Angle GFL; and BA is to AE, as GF is to FL: Therefore the two Triangles BAE, GFL, have one Angle of the one equal to one Angle of the other, viz. the Angle BAE equal to the Angle GFL, and the Sides about the equal Angles proportional. Wherefore the Triangle ABE is * equiangular to the Triangle FGL; * 6. 6.
and

- and so the Angle AEB is equal to the Angle FLG :
 * 21. 3. But the Angle AEB is * equal to the Angle AMB ,
 for they stand on the same Circumference; and the
 Angle FLG is † equal to the Angle FNG : There-
 † 31. 3. fore the Angle AMB is equal to the Angle FNG .
 But the right Angle BAM is † equal to the right
 Angle GFN ; wherefore the other Angle shall be
 equal to the other Angle: And so the Triangle AMB
 is equiangular to the Triangle FGN ; and, conse-
 * 4. 6. quently, * as BM is to GN , so is BA to GF . But
 the Proportion of the Square of BM to the Square
 of GN , is duplicate of the Proportion of BM to
 † 20. 6. GN ; and the Proportion of the Polygon $ABCDE$
 to the Polygon $FGHKL$ is † duplicate of the Pro-
 portion of BA to GF : Wherefore, as the Square of
 BM is to the Square of GN , so is the Polygon
 $ABCDE$ to the Polygon $FGHKL$. Therefore
*similar Polygons, inscribed in Circles, are to one an-
 other as the Squares of the Diameters of the Circles;*
 which was to be demonstrated.

L E M M A.

*If there be two unequal Mag-
 nitudes propos'd, and from
 the greater be taken a Part
 greater than its Half; and
 if from what remains there
 be again taken a Part
 greater than half this Re-
 mainder; and again from
 this last Remainder a Part
 greater than its Half; and
 if this be done continually,
 there will remain at last a
 Magnitude that shall be
 less than the lesser of the
 propos'd Magnitudes.*



LET AB and C be two unequal Magnitudes,
 whereof AB is the greater: I say, if from AB
 be taken a greater Part than half, and from the Part
 remaining

remaining there be again taken a Part greater than its Half, and this be done continually, there will remain a Magnitude, at last, that shall be less than the Magnitude C.

For C, being some Number of Times multiplied, will become greater than the Magnitude AB. Let it be multiplied, and let DE be a Multiple of C greater than AB; divide DE into Parts DF, FG, GE, each equal to C; and take BH, a Part greater than half of AB, from AB, and again from AH the Part HK greater than half AH, and from AK a Part greater than half AK, and so on, until the Divisions, that are in AB, are equal in Number to the Divisions in DE: Therefore let the Divisions AK, KH, HB, be equal in Number to the Divisions DF, FG, GE. Then, because DE is greater than AB, and the Part EG is taken from ED, being less than half thereof, and the Part BH, greater than half of AB, is taken from it, the Part remaining, GD, shall be greater than the Part remaining HA. Again, because GD is greater than HA; and GF, being half of GD, is taken from the same; and HK, being greater than half HA, is taken from this likewise; the Part remaining, FD, shall be greater than the Part remaining AK: But FD is equal to C; therefore C is greater than AK; and so the Magnitude AK is less than C: Therefore the Magnitude AK, being the Part remaining of the Magnitude AB, is less than the lesser propos'd Magnitude C; *which was to be demonstrated.* If the Halves of the Magnitudes should have been taken, we demonstrate this after the same manner. *This is the first Proposition of the tenth Book.*

PROPOSITION II.

THEOREM.

Circles are to each other as the Squares of their Diameters.

LET ABCD, EFGH, be Circles, whose Diameters are BD, FH. I say, as the Square of BD is to the Square of FH, so the Circle ABCD is to the Circle EFGH. For,

For, if it be not so, the Square of BD shall be to the Square of FH , as the Circle $ABCD$ is to some Space either less or greater than the Circle $EFGH$. First, let it be to a Space S , less than the Circle $EFGH$, and let the Square $EFGH$ be describ'd therein.

This Square $EFGH$ will be greater than half the Circle $EFGH$; because, if we draw Tangents to the Circle thro' the Points E, F, G, H , the Square $EFGH$ will be half that describ'd about the Circle: But the Circle is less than the Square describ'd about it; therefore the Square $EFGH$ is greater than half the Circle $EFGH$. Let the Circumferences EF, FG, GH, HE , be bisected in the Points K, L, M, N ; and join $EK, KF, FL, LG, GM, MH, HN, NE$: Then each of the Triangles EKF, FLG, GMH, HNE , will be * greater than one half of the Segment of the Circle it stands in; because, if Tangents at the Circle be drawn thro' the Points K, L, M, N , and the Parallelograms that are on the right Lines EF, FG, GH, HE , be completed, each of the Triangles EKF, FLG, GMH, HNE , is half of each of the corresponding Parallelograms: But the Segment is less than the Parallelogram; wherefore each of the Triangles EKF, FLG, GMH, HNE , is greater than one half of the Segment of the Circle in which it stands: Therefore, if these Circumferences be again bisected, and right Lines be drawn joining the Points of Bisection, and you do thus continually, there will at last remain Segments of the Circle, that shall be less than the Excess by which the Circle $EFGH$ exceeds the Space S . For it is demonstrated, in the foregoing *Lemma*, that if there be two unequal Magnitudes propos'd, and if from the greater a Part greater than half be taken from it, and again from the Part remaining a Part greater than half be taken, and you do this continually, there will at last remain a Magnitude that will be less than the lesser propos'd Magnitude. Let the Segments of the Circle $EFGH$, on the right Lines $EK, KF, FL, LG, GM, MH, HN, NE$, be those which are less than the Excess whereby the Circle $EFGH$ exceeds the Space S ; and then the remaining Polygon $EKF LG MH N$ shall be greater than the Space S . Also, describe the Polygon $AXBOCPDR$ in the Circle $ABCD$, similar to the Polygon

* 41. 1.

Polygon EKFLGMHN. Wherefore, as the Square of BD is to the Square of FH, so is the Polygon AXBOCPDR to * the Polygon EKFLGMHN. * *I of this.*
 But as the Square of BD is to the Square of FH, so is the Circle ABCD to the Space S: Wherefore, as the Circle ABCD is to the Space S, so is † the Polygon † *II. 5.*
 AXBOCPDR to the Polygon EKFDGMHN.
 But the Circle ABCD is greater than the Polygon in it; wherefore the Space S shall be † also greater than the Polygon EKFLGMHN: But it is less †, † *From the Hyp.*
 likewise, which is absurd; therefore the Square of BD to the Square of FH, is not as the Circle ABCD to some Space less than the Circle EFGH. After the same manner we likewise demonstrate, that the Square of FH to the Square of BD is not as the Circle EFGH to some Space less than the Circle ABCD. Lastly, I say, the Square of BD to the Square of FH, is not as the Circle ABCD to some Space greater than the Circle EFGH; for, if it be possible, let it be so, and let the Space S be greater than the Circle EFGH: Then it shall be (by Inversions) as the Square of FH is to the Square of BD, so is the Space S to the Circle ABCD. But because S is greater than the Circle EFGH, the Space shall be to the Circle ABCD, as the Circle EFGH is to some Space less than the Circle ABCD: Therefore, as the Square of FH is to the Square of BD, so is * the Circle * *II. 5.*
 EFGH to some Space less than the Circle ABCD, which has been demonstrated to be impossible; wherefore the Square of BD to the Square of FH, is not as the Circle ABCD to some Space greater than the Circle EFGH: But this also has been prov'd, that the Square of BD to the Square of FH, is not as the Circle ABCD to some Space less than the Circle EFGH: Wherefore, as the Square of BD is to the Square of FH, so shall the Circle ABCD be to the Circle EFGH. Wherefore, Circles are to each other as the Squares of their Diameters; *which was to be demonstrated.*

PROPOSITION III.

THEOREM.

Every Pyramid, having a triangular Base, may be divided into two Pyramids, equal and similar to one another, having triangular Bases, and similar to the whole Pyramid, and into two equal Prisms, which two Prisms are greater than the Half of the whole Pyramid.

LET there be a Pyramid, whose Base is the Triangle ABC , and Vertex the Point D . I say, the Pyramid $ABCD$ may be divided into two Pyramids equal and similar to one another, having triangular Bases, and similar to the Whole; and into two equal Prisms, which two Prisms are greater than the Half of the whole Pyramid.

For, bisect AB , BC , CA , AD , DB , DC , in the Points E , F , G , H , K , L ; and join EH , EG , GH , HK , KL , LH , EK , KF , FG : Then, because AE is equal to EB , and AH to HD , EH shall be * parallel to DB : For the same Reason, HK also is parallel to AB ; therefore $HEBK$ is a Parallelogram; and so HK is † equal to EB : But EB is equal to AE ; therefore AE shall be also equal to HK : But AH is equal to HD ; wherefore the two Sides AE , AH , are equal to the two Sides KH , HD , each to each, and the Angle EAH is ‡ equal to the Angle KHD ; wherefore the Base EH is * equal to the Base KD ; and so the Triangle AEH is equal and similar to the Triangle KHD . For the same Reason, the Triangle AHG shall also be equal and similar to the Triangle $HL D$; and because the two right Lines EH , HG , touching each other, are parallel to the two right Lines KD , DL , touching each other, and not in the same Plane with them, they shall contain † equal Angles. Therefore the Angle EHG is equal to the Angle KDL . Again, because the two Sides EH , HG , are equal to the two Sides KD , DL , each to each, and the Angle EHG is equal to the Angle KDL , the Base EG shall be * equal to the Base KL ; and therefore the Triangle EHG is equal and similar to the Triangle KDL . For the same Reason, the Tri-

angle

* 2. 6.

† 34. 1.

‡ 29. 1.

* 4. 1.

† 10. 11.

* 4. 1.

angle AEG is also equal and similar to the Triangle HKL; wherefore the Pyramid whose Base is the Triangle AEG, and Vertex the Point H, is equal and similar to the Pyramid whose Base is the Triangle HKL, and Vertex the Point D. And because HK is drawn parallel to the Side AB of the Triangle ADB, the Triangle ADB shall be equiangular to the Triangle DKH, and they have their Sides proportional; therefore the Triangle ADB is similar to the Triangle DHK. And, for the same Reason, the Triangle DBC is similar to the Triangle DKL; and the Triangle AHG to the Triangle DHL. And since the two right Lines BA, AC, touching each other, are parallel to the two Lines KH, HL, touching each other, not being in the same Plane with them, these shall contain equal Angles; therefore the Angle BAC is equal to the Angle KHL. And BA is to AC, as KH is to HL; wherefore the Triangle ABC is similar to the Triangle HKL; and so the Pyramid, whose Base is the Triangle ABC, and Vertex the Point D, is similar to the Pyramid, whose Base is the Triangle HKL, and Vertex the Point D. But the Pyramid, whose Base is the Triangle HKL, and Vertex the Point D, has been prov'd similar to the Pyramid whose Base is the Triangle AEG, and Vertex the Point H; therefore the Pyramid whose Base is the Triangle ABC, and Vertex the Point D, is similar to the Pyramid whose Base is the Triangle AEG, and Vertex the Point H: Wherefore both the Pyramids AEGH, HKLD, are similar to the whole Pyramid ABCD. And because BF is equal to FC, the Parallelogram EBF G will be double to the Triangle GFC. And since there are two Prisms of equal Altitude, one of which has that Parallelogram for a Base, and the other the Triangle, and the Parallelogram is double to the Triangle; those Prisms will be † equal to one another: Therefore the Prism con-† 40. 11.
tain'd under the two Triangles BKF, EHG, and the three Parallelograms EBF G, EBKH, KHGF, is equal to the Prism contain'd under the two Triangles GFC, HKL, and the three Parallelograms KFCL, LCGH, HKFG. And it is manifest, that each of those Prisms, the Base of one of which is the Parallelogram EBF G, and opposite Base to that the right Line
R K H,

KH , and the Base of the other Triangle GFC , and the opposite Base to this, the Triangle KLH , are greater than either of the Pyramids, whose Bases are the Triangles AEG , HKL , and Vertices the Points H and D . For since, if the right Lines EF , EH , be join'd, the Prism, whose Base is the Parallelogram $EBFG$, and opposite Base to that the right Line KH , is greater than the Pyramid, whose Base is the Triangle EBF , and Vertex the Point K . But the Pyramid whose Base is the Triangle EBF , and Vertex the Point K , is equal to the Pyramid whose Base is the Triangle AEG , and Vertex the Point H ; for they are contain'd under equal and similar Planes: Wherefore the Prism whose Base is the Parallelogram $EBFG$, and opposite Base to it the right Line HK , is greater than the Pyramid whose Base is the Triangle AEG , and Vertex the Point H . But the Prism whose Base is the Parallelogram $EBFG$, and opposite Base to it the right Line HK , is equal to the Prism whose Base is the Triangle GFC , and opposite Base to this the Triangle HKL ; and the Pyramid whose Base is the Triangle AEG , and Vertex the Point H , is equal to the Pyramid whose Base is the Triangle HKL , and Vertex the Point D : Therefore the two Prisms aforesaid are greater than the said two Pyramids, whose Bases are the Triangles AEG , HKL , and Vertices the Points H , D : And so the whole Pyramid whose Base is the Triangle ABC , and Vertex the Point D , is divided into two equal Pyramids, similar to each other, and to the Whole; and into two equal Prisms, which two Prisms, together, are greater than half of the whole Pyramid. Therefore *every Pyramid, having a triangular Base, may be divided into two Pyramids, equal and similar to one another, having triangular Bases, and similar to the whole Pyramid; and into two equal Prisms, which two Prisms are greater than the Half of the whole Pyramid; which was to be demonstrated.*

PROPOSITION IV.

THEOREM.

If there are two Pyramids of the same Altitude, having triangular Bases, and each of them be divided into two Pyramids, equal to one another, and similar to the Whole, as also into two equal Prisms; and if, in like manner, each of the two Pyramids, made by the former Division, be divided, and this be done continually; then, as the Base of one Pyramid is to the Base of the other Pyramid, so are all the Prisms that are in one Pyramid, to all the Prisms that are in the other Pyramid, being equal in Multitude.

LET there be two Pyramids of the same Altitude, having the triangular Bases ABC , DEF , whose Vertices are the Points G , H ; and let each of them be divided into two Pyramids, equal to one another, and similar to the Whole, and into two equal Prisms; and if, in like manner, each of the Pyramids, made by the former Division, be conceiv'd to be divided, and this be done continually; I say, as the Base ABC is to the Base DEF , so are all the Prisms that are in the Pyramid $ABCG$, to all the Prisms that are in the Pyramid $DEFH$, being equal in Multitude.

For, since BX is equal to XC , and AL to LC ; XL shall be * parallel to AB , and the Triangle ABC * 2. 6. similar to the Triangle LXC . For the same Reason, the Triangle DEF shall be also similar to the Triangle RQF ; and because BC is double to CX , and EF to FQ ; it shall be, as BC is to CX , so is EF to FQ : And since there are describ'd upon BC , CX , right-lin'd Figures ABC , LXC , similar and alike situate, and upon EF , FQ , right-lin'd Figures DEF , RQF , similar and alike situate; therefore, as the Triangle BAC is to the Triangle LXC , so is † the † 22. 6. Triangle DEF to the Triangle RQF ; and (by Alternation) as the Triangle ABC is to the Triangle DEF , so is the Triangle LXC to the Triangle RQF . But as the Triangle LXC is to the Triangle RQF , so is ‡ the Prism, whose Base is the Triangle LXC , † 28. and and opposite Base to that the Triangle OMN , to the ‡ 2. 11. Prism,

R 2

• 11. 5.

Prism, whose Base is the Triangle RQF , and opposite Base to that the Triangle STY ; therefore, as the Triangle ABC is to the Triangle DEF , so is * the Prism whose Base is the Triangle LXC , and opposite Base to that the Triangle OMN , to the Prism whose Base is the Triangle RQF , and opposite Base to that the Triangle STY : And because the two Prisms that are in the Pyramid $ABCG$ are equal to one another, as also those two that are in the Pyramid $DEFH$; it shall be, as the Prism whose Base is the Parallelogram $KLXB$, and opposite Base to that the right Line MO , is to the Prism whose Base is the Triangle LXC , and opposite Base to that the Triangle OMN , so is the Prism whose Base is the Parallelogram $EPRQ$, and opposite Base to that the right Line ST , to the Prism whose Base is the Triangle RQF , and opposite Base to that the Triangle STY : Therefore, (by compounding) as the Prisms $KBXLMO$, $LXCMNO$, to the Prism $LXCMNO$, so the Prisms $PEQRST$, $RQFSTY$, to the Prism $RQFSTY$: And (by Alternation) as the Prisms $KBXLMO$, $LXCMNO$, to the Prisms $PEQRST$, $RQFSTY$, so the Prism $LXCMNO$, to the Prism $RQFSTY$: But as the Prism $LXCMNO$ is to the Prism $RQFSTY$, so has the Base LXC been prov'd to be to the Base RFQ ; and so the Base ABC to the Base DEF : Therefore, also, as the Triangle ABC is to the Triangle DEF , so are the two Prisms that are in the Pyramid $ABCG$, to the two Prisms that are in the Pyramid $DEFH$. If, in the same manner, each of the Pyramids $OMNG$, $STYH$, made by the former Division, be divided, it shall be, as the Base OMN is to the Base STY , so the two Prisms that are in the Pyramid $OMNG$, to the two Prisms that are in the Pyramid $STYH$. But as the Base OMN is to the Base STY , so is the Base ABC to the Base DEF : Therefore, as the Base ABC is to the Base DEF , so are the two Prisms that are in the Pyramid $ABCG$, to the two Prisms that are in the Pyramid $DEFH$; and so the two Prisms that are in the Pyramid $OMNG$, to the two Prisms that are in the Pyramid $STYH$; and so the four to the four. We demonstrate the same of Prisms made by the Division of the Pyramids $AKLO$, $DPRS$,
and

and of all other Prisms, being equal in Multitude ;
which was to be demonstrated.

PROPOSITION V.

THEOREM.

Pyramids of the same Altitude, and having triangular Bases, are to one another as their Bases.

LET there be two Pyramids of the same Altitude, having the triangular Bases ABC , DEF , whose Vertices are the Points G , H . I say, as the Base ABC is to the Base DEF , so is the Pyramid $ABCG$ to the Pyramid $DEFH$.

For, if it be not so, then it shall be, as the Base ABC is to the Base DEF , so is the Pyramid $ABCG$ to some Solid greater or less than the Pyramid $DEFH$. First, let it be to a Solid less, which let be Z , and divide the Pyramid $DEFH$ into two Pyramids equal to each other, and similar to the Whole, and into two equal Prisms ; then these two Prisms are greater than the half of the whole Pyramid : And, again, let the Pyramids, made by the former Division, be divided after the same manner, and let this be done continually, until the Pyramids in the Pyramid $DEFH$ are less than the Excess by which the Pyramid $DEFH$ exceeds the Solid Z . Let these, for Example, be the Pyramids $DPRS$, $STYH$; then the Prisms remaining in the Pyramid $DEFH$, are greater than the Solid Z : Also, let the Pyramid $ABCG$ be divided into the same Number of similar Parts as the Pyramid $DEFH$ is ; and then, as the Base ABC is to the Base DEF , so * the Prisms that are in the * 4 of this. Pyramid $ABCG$, to the Prisms that are in the Pyramid $DEFH$. But as the Base ABC is to the Base DEF , so is the Pyramid $ABCG$ to the Solid Z ; and therefore, as the Pyramid $ABCG$ is to the Solid Z , so are the Prisms that are in the Pyramid $ABCG$, to the Prisms that are in the Pyramid $DEFH$. But the Pyramid $ABCG$ is greater than the Prisms that are in it ; wherefore, also, the Solid Z is greater than the Prisms that are in the Pyramid $DEFH$. But

* From what
has been al-
ready demon-
strated.

it is less, * also, which is absurd ; therefore the Base ABC to the Base DEF, is not as the Pyramid ABCG to some Solid less than the Pyramid DEFH. After the same manner we demonstrate, that the Base DEF to the Base ABC, is not as the Pyramid DEFH to some Solid less than the Pyramid ABCG : Therefore, I say, neither is the Base ABC to the Base DEF, as the Pyramid ABCG to some Solid greater than the Pyramid DEFH. For, if this is possible, let it be to the Solid I, greater than the Pyramid DEFH ; then (by Inversion) the Base DEF shall be to the Base ABC, as the Solid I to the Pyramid ABCG : But since the Solid I is greater than the Pyramid DEFH, it shall be, as the Solid I is to the Pyramid ABCG, so is the Pyramid DEFH to some Solid less than the Pyramid ABCG, as just now has been prov'd ; and so, as the Base DEF is to the Base ABC, so is the Pyramid DEFH to some Solid less than the Pyramid ABCG, which is absurd : Therefore the Base ABC to the Base DEF, is not as the Pyramid ABCG to some Solid greater than the Pyramid DEFH. But it has been also prov'd, that the Base ABC to the Base DEF, is not as the Pyramid ABCG to some Solid less than the Pyramid DEFH ; wherefore, as the Base ABC is to the Base DEF, so is the Pyramid ABCG to the Pyramid DEFH. Therefore *Pyramids of the same Altitude, and having triangular Bases, are to one another as their Bases* ; which was to be demonstrated.

PROPOSITION VI.

THEOREM.

Pyramids of the same Altitude, and having polygonous Bases, are to one another as their Bases.

LET there be Pyramids of the same Altitude, which have the polygonous Bases ABCDE, FGHLK, and let their Vertices be the Points M, N. I say, as the Base ABCDE is to the Base FGHLK, so is the Pyramid ABCDEM to the Pyramid FGHLKN.

For,

For, let the Base $ABCDE$ be divided into the Triangles ABC , ACD , ADE ; and the Base $FGHKL$ into the Triangles FGH , CHK , FKL ; and let Pyramids be conceiv'd upon every one of those Triangles of the same Altitude with the Pyramids ABC , DEM , $FGHKLN$: Then, because the Triangle ABC is to the Triangle ACD , as * the Pyramid $ABCM$ is to the Pyramid $ACDM$; and (by compounding) as the Trapezium $ABCD$ is to the Triangle ACD , so is the Pyramid $ABCDM$ to the Pyramid $ACDM$: But as the Triangle ACD is to the Triangle ADE , so is * the Pyramid $ACDM$ to the Pyramid $ADEM$. Wherefore, (by Equality of Proportion) as the Base $ABCD$ is to the Base ADE , so is the Pyramid $ABCDM$ to the Pyramid $ADEM$: And again, (by Composition of Proportion) as the Base $ABCDE$ is to the Base ADE , so is the Pyramid $ABCDEM$ to the Pyramid $ADEM$. For the same Reason, as the Base $FGHKL$ is to the Base FKL , so is the Pyramid $FGHKLN$ to the Pyramid $FKLN$: And since there are two Pyramids $ADEM$, $FKLN$, having triangular Bases, and the same Altitude; the Base ADE shall be * to the Base FKL , as the Pyramid $ADEM$ to the Pyramid $FKLN$: And since the Base $ABCDE$ is to the Base ADE , as the Pyramid $ABCDEM$ is to the Pyramid $ADEM$; and as the Base ADE is to the Base FKL , so is the Pyramid $ADEM$ to the Pyramid $FKLN$; it shall be, (by Equality of Proportion) as the Base $ABCDE$ to the Base FKL , so is the Pyramid $ABCDEM$ to the Pyramid $FKLN$: But as the Base FKL is to the Base $FGHKL$, so was the Pyramid $FKLN$ to the Pyramid $FGHKLN$. Wherefore, again, (by Equality of Proportion) as the Base $ABCDE$ is to the Base $FGHKL$, so is the Pyramid $ABCDEM$ to the Pyramid $FGHKLN$. Therefore *Pyramids of the same Altitude, and having polygonous Bases, are to one another as their Bases*; which was to be demonstrated.

PROPOSITION VII.

THEOREM.

Every Prism, having a triangular Base, may be divided into three Pyramids equal to one another, and having triangular Bases.

LET there be a Prism whose Base is the Triangle ABC , and opposite Base to that the Triangle DEF . I say, the Prism $ABCDEF$ may be divided into the three equal Pyramids, that have triangular Bases.

For, join BD , EC , CD : Then, because $ABED$ is a Parallelogram, whose Diameter is BD , the Triangle ABD shall be * equal to the Triangle EBD . Therefore the Pyramid whose Base is the Triangle ABD , and Vertex the Point C , is † equal to the Pyramid whose Base is the Triangle EDB , and Vertex the Point C . But the Pyramid whose Base is the Triangle EDB , and Vertex the Point C , is the same as the Pyramid whose Base is the Triangle ECB , and Vertex the Point D ; for they are contain'd under the same Planes: Therefore the Pyramid, whose Base is the Triangle ABD , and Vertex the Point C , is equal to the Pyramid whose Base is the Triangle ECB , and Vertex the Point D . Again, because $FCBE$ is a Parallelogram, whose Diameter is CE , the Triangle ECF shall be * equal to the Triangle CBE ; and so the Pyramid whose Base is the Triangle BEC , and Vertex the Point D , is † equal to the Pyramid, whose Base is the Triangle ECF , and Vertex the Point D . But the Pyramid whose Base is the Triangle BCE , and Vertex the Point D , has been prov'd equal to the Pyramid whose Base is the Triangle ABD , and Vertex the Point C : Wherefore, also, the Pyramid, whose Base is the Triangle CEF , and Vertex the Point D , is equal to the Pyramid, whose Base is the Triangle ABD , and Vertex the Point C ; therefore the Prism $ABCDEF$ is divided into three Pyramids equal to one another, and having triangular Bases. And because the Pyramid, whose Base is the Triangle ABD , and Vertex the Point C , is the same with the Pyramid whose Base is the Triangle CAB , and Vertex the Point D ; for they

* 34. 1.

† 6 of this.

they are contain'd under the same Planes ; and the Pyramid, whose Base is the Triangle ABD , and Vertex the Point C , has been prov'd to be a third Part of the Prism, whose Base is the Triangle ABC , and opposite Base to that the Triangle DEF : Therefore, also, the Pyramid, whose Base is the Triangle ABC , and Vertex the Point D , is a third Part of the Prism having the same Base, *viz.* the Triangle ABC , and the opposite Base the Triangle DEF ; *which was to be demonstrated.*

Coroll. 1. It is manifest, from hence, that every Pyramid is a third Part of a Prism, having the same Base, and an equal Altitude ; because, if the Base of a Prism, as also the opposite Base, be of any other Figure, it may be divided into Prisms having triangular Bases.

2. Prisms of the same Altitude are to one another as their Bases.

PROPOSITION VIII.

THEOREM.

Similar Pyramids, having triangular Bases, are in a triplicate Proportion of their homologous Sides.

LET there be two Pyramids similar and alike situate, having the triangular Bases ABC , DEF ; and let their Vertices be the Points G , H . I say, the Pyramid $ABCG$ to the Pyramid $DEFH$, has a Proportion triplicate of that which BC has to EF .

For, complete the solid Parallelepipedons $BGML$, $EHPO$; then, because the Pyramid $ABCG$ is similar to the Pyramid $DEFH$, the Angle ABC shall be * equal to the Angle DEF , the Angle GBC * *Def. 9. 11.* equal to the Angle HEF , and the Angle ABG equal to the Angle DEH : And AB is to DE , as BC is to EF ; and so is BG to EH . Therefore, because AB is to DE , as BC is to EF ; and the Sides about the equal Angles are proportional ; the Parallelogram BM shall † be similar to the Parallelo- † 6. 6. gram EP . For the same Reason, the Parallelogram BN is similar to the Parallelogram ER , and the Parallelogram

rallelogram BK to the Parallelogram EX. Therefore three Parallelograms BM, KB, BN, are similar to three Parallelograms EP, EX, ER. But the three MB, BK, BN, are equal and similar to the three opposite ones; as also the three EP, EX, ER: Therefore the Solids BGML, EHPO, are contain'd under equal Numbers of similar and equal Planes; and, consequently, the Solid BGML is similar to the Solid EHPO. But similar solid Paralelepipedons are * to each other in a triplicate Proportion of their homologous Sides; therefore the Solid BGML to the Solid EHPO, has a Proportion triplicate of that which the homologous Side BC has to the homologous Side EF. But as the Solid BCM L is to the Solid EHPO, so is † the Pyramid ABCG to the Pyramid DEFH; for the Pyramid is the one sixth Part of that Solid, since the Prism, which is the half of the solid Paralelepipedon, is triple of the Pyramid. Wherefore the Pyramid ABCG to the Pyramid DEFH, shall have a triplicate Proportion to that which BC has to EF; *which was to be demonstrated.*

* 33. 11.

† 15. 5.

Coroll. From hence it is manifest, that similar Pyramids, having polygonous Bases, are to one another in a triplicate Proportion of their homologous Sides. For, if they be divided into Pyramids having triangular Bases; because their similar polygonous Bases are divided into similar Triangles equal in Number, and homologous to the Wholes; it shall be, as one Pyramid, having a triangular Base in one of the Pyramids, is to a Pyramid having a triangular Base in the other Pyramid; so are all the Pyramids, having triangular Bases in one Pyramid, to all the Pyramids having triangular Bases in the other Pyramid; that is, so is one of the Pyramids, having the polygonous Base, to the other: But a Pyramid, having a triangular Base to a Pyramid having a triangular Base, is in a triplicate Proportion of the homologous Sides. Therefore one Pyramid, having a polygonous Base to another Pyramid having a similar Base, is in a triplicate Proportion of their homologous Sides.

PROPOSITION IX.

THEOREM.

The Bases and Altitudes of equal Pyramids, having triangular Bases, are reciprocally proportional; and those Pyramids, having triangular Bases, whose Bases and Altitudes are reciprocally proportional, are equal.

LET there be equal Pyramids, having the triangular Bases ABC , DEF , and Vertices the Points G , H . I say, the Bases and Altitudes of the Pyramids $ABCG$, $DEFH$, are reciprocally proportional; that is, as the Base ABC is to the Base DEF , so is the Altitude of the Pyramid $DEFH$ to the Altitude of the Pyramid $ABCG$.

For, complete the solid Parallelepipedons $BGML$, $EHPO$; then, because the Pyramid $ABCG$ is equal to the Pyramid $DEFH$; and the Solid $BGML$ is sextuple, the Pyramid $ABCG$, and the Solid $EHPO$, sextuple of the Solid $DEFH$, the Solid $BGML$ shall be * equal to the Solid $EHPO$. But the Bases * 15. 5. and Altitudes of equal solid Parallelepipedons are reciprocally proportional; therefore, as the Base BM is to the Base EP , so is † the Altitude of the Solid † 34. II. $EHPO$ to the Altitude of the Solid $BGML$. But as the Base BM is to the Base EP , so is † the Triangle ABC to the Triangle DEF ; therefore, as the Triangle ABC is to the Triangle DEF , so is the Altitude of the Solid $EHPO$ to the Altitude of the Solid $BGML$. But the Altitude of the Solid $EHPO$ is the same as the Altitude of the Pyramid $DEFH$; and the Altitude of the Solid $BGML$ the same as the Altitude of the Pyramid $ABCG$; therefore, as the Base ABC is to the Base DEF , so is the Altitude of the Pyramid $DEFH$ to the Altitude of the Pyramid $ABCG$: Wherefore, the Bases and Altitudes of the equal Pyramids $ABCG$, $DEFH$, are reciprocally proportional; and if the Bases and Altitudes of the Pyramids $ABCG$, $DEFH$, are reciprocally proportional, that is, if the Base ABC to the Base DEF , be as the Altitude of the Pyramid $DEFH$,

DEFH to the Altitude of the Pyramid ABCG; I say, the Pyramid ABCG is equal to the Pyramid DEFH: For, the same Construction remaining, because the Base ABC to the Base DEF, is as the Altitude of the Pyramid DEFH to the Altitude of the Pyramid ABCG; and as the Base ABC is to the Base DEF, so is the Parallelogram BM to the Parallelogram EP; the Parallelogram BM to the Parallelogram EP shall be also as the Altitude of the Pyramid DEFH is to the Altitude of the Pyramid ABCG. But as the Altitude of the Pyramid DEFH is the same as the Altitude of the solid Parallelepipedon EHPO, and the Altitude of the Pyramid ABCG the same as the Altitude of the solid Parallelepipedon BGML; therefore the Base BM to the Base EP will be as the Altitude of the solid Parallelepipedon EHPO to the Altitude of the solid Parallelepipedon BGML. But those solid Parallelepipedons whose Bases and Altitudes are reciprocally proportional, are † equal to each other; therefore the solid Parallelepipedon BGML is equal to the solid Parallelepipedon EHPO; and the Pyramid ABCG is a sixth Part of the Solid BGML: And, in like manner, the Pyramid DEFH is a sixth Part of the Solid EHPO; therefore the Pyramid ABCG is equal to the Pyramid DEFH. Wherefore *the Bases and Altitudes of equal Pyramids, having triangular Bases, are reciprocally proportional; and those Pyramids, having triangular Bases, whose Bases and Altitudes are reciprocally proportional, are equal; which was to be demonstrated.*

† 34. 11.

PROPOSITION X.

THEOREM.

Every Cone is a third Part of a Cylinder, having the same Base, and an equal Altitude.

LET a Cone have the same Base as a Cylinder, viz. the Circle ABCD, and an Altitude equal to it. I say, the Cone is a third Part of the Cylinder; that is, the Cylinder is triple to the Cone.

For,

For, if the Cylinder be not triple to the Cone, it shall be greater or less than triple thereof. First, let it be greater than triple to the Cone, and let the Square ABCD be describ'd in the Circle ABCD; then the Square ABCD is greater than one half of the Circle ABCD. Now let a Prism be erected upon the Square ABCD, having the same Altitude as the Cylinder, and this Prism will be greater than one half of the Cylinder; because, if a Square be circumscrib'd about the Circle ABCD, the inscrib'd Square will be one half of the circumscrib'd Square; and if a Prism be erected upon the circumscrib'd Square of the same Altitude as the Cylinder, since Prisms are * to one * 2 Cor. 7. of this. another as their Bases, the Prism erected upon the Square ABCD is one half of the Prism erected upon the Square describ'd about the Circle ABCD. But the Cylinder is lesser than the Prism erected on the Square describ'd about the Circle ABCD; therefore the Prism erected on the Square ABCD, having the same Height as the Cylinder, is greater than one half of the Cylinder. Let the Circumferences AB, BC, CD, DA, be bisected in the Points E, F, G, H; and join AE, EB, BF, FC, CG, GD, DH, HA: Then each of the Triangles AEB, BFC, CGD, DHA, is † greater than the half of each of the Seg- † This follows from 2. of this. ments in which they stand. Let Prisms be erected from each of the Triangles AEB, BFC, CGD, DHA, of the same Altitude as the Cylinder; then every one of these Prisms erected is greater than its correspondent Segment of the Cylinder. For, because, if Parallels be drawn through the Points E, F, G, H, to AB, BC, CD, DA, and Parallelograms be completed on the said AB, BC, CD, DA, on which are erected solid Parallelepipedons of the same Altitude as the Cylinder; then each of those Prisms that are on the Triangles AEB, BFC, CGD, DHA, are Halves † of each of the solid Parallelepipedons; and the Segments of the Cylinder are less than the erected solid Parallelepipedons; and, consequently, the Prisms that are on the Triangles AEB, BFC, CGD, DHA, are greater than the Halves of the Segments of the Cylinder; and so, bisecting the other Circumferences, joining right Lines, and on every one of the Triangles erecting Prisms of the same Height as the Cylinder;

Cylinder; and, doing this continually, we shall at last have certain Portions of the Cylinder left, that are less than the Excess by which the Cylinder exceeds triple the Cone.

* 1 Cor. 7.
of this.

Now let these Portions remaining be AE, EB, BF, FC, CG, GD, DH, HA; then the Prism remaining, whose Base is the Polygon AEBFCGDH, and Altitude equal to that of the Cylinder's, is greater than the Triple of the Cone. But the Prism whose Base is the Polygon AEBFCGDH, and Altitude the same as that of the Cylinder's, is * triple of the Pyramid whose Base is the Polygon AEBFCGDH, and Vertex the same as that of the Cone; and therefore the Pyramid whose Base is the Polygon AEBFCGDH, and Vertex the same as that of the Cone, is greater than the Cone whose Base is the Circle ABCD: But it is lesser also (for it is comprehended by it), which is absurd; therefore the Cylinder is not greater than triple the Cone. I say, it is neither lesser than triple the Cone: For, if it be possible, let the Cylinder be less than triple the Cone; then (by Inversion) the Cone shall be greater than a third Part of the Cylinder: Let the Square ABCD be describ'd in the Circle ABCD; then the Square ABCD is greater than half of the Circle ABCD: And let a Pyramid be erected on the Square ABCD, having the same Vertex as the Cone; then the Pyramid erected is greater than one half of the Cone; because, as has been already demonstrated, if a Square be describ'd about the Circle, the Square ABCD shall be half thereof: And if solid Parallelepipedons be erected upon the Squares of the same Altitude as the Cones, which are also call'd Prisms; then the Prism erected on the Square ABCD is one half of that erected on the Square describ'd about the Circle; for they are to each other as their Bases, and so likewise are their third Parts: Therefore the Pyramid whose Base is the Square ABCD, is one half of that Pyramid erected upon the Square describ'd about the Circle. But the Pyramid erected upon the Square describ'd about the Circle, is greater than the Cone, for it comprehends it; therefore the Pyramid whose Base is the Square ABCD, and Vertex the same as that of the Cone, is greater than one half

of

of the Cone. Bisect the Circumferences AB , BC , CD , DA , in the Points E , F , G , H ; and join AE , EB , BF , FC , CG , GD , DH , HA ; and then each of the Triangles AEB , BFC , CGD , DHA , is greater than one Half of each of the Segments they are in. Let Pyramids be erected upon each of the Triangles AEB , BFC , CGD , DHA , having the same Vertex as the Cone; then each of these Pyramids, thus erected, is greater than one half of the Segment of the Cone in which it is; and so bisecting the remaining Circumferences, joining the right Lines, and erecting Pyramids upon every of the Triangles having the same Altitude as the Cone, and doing this continually, we shall at last have Segments of the Cone left, that will be less than the Excess by which the Cone exceeds the one third Part of the Cylinder: Let these Segments be those that are on AE , EB , BF , FC , CG , GD , DH , HA ; and then the remaining Pyramid, whose Base is the Polygon $AEBFCGDH$, and Vertex the same as that of the Cone, is greater than a third Part of the Cylinder: But the Pyramid whose Base is the Polygon $AEBFCGDH$, and Vertex the same as that of the Cone, is one third Part of the Prism whose Base is the Polygon $AEBFCGDH$, and Altitude the same as that of the Cylinder: Therefore the Prism, whose Base is the Polygon $AEBFCGDH$, and Altitude the same as that of the Cylinder, is greater than the Cylinder, whose Base is the Circle $ABCD$; but it is less also (as being comprehended thereby); which is absurd; therefore the Cylinder is not less than triple of the Cone: But it has been prov'd also not to be greater than triple of the Cone; therefore the Cylinder is necessarily triple of the Cone. Wherefore *every Cone is a third Part of a Cylinder, having the same Base, and an equal Altitude*; which was to be demonstrated.

PROPOSITION XI.

THEOREM.

Cones and Cylinders, of the same Altitude, are to one another as their Bases.

LET there be Cones and Cylinders of the same Altitude, whose Bases are the Circles $ABCD$, $EFGH$, Axes KL , MN , and Diameters of the Bases AC , EG . I say, as the Circle $ABCD$ is to the Circle $EFGH$, so is the Cone AL to the Cone EN .

For, if it be not so, it shall be, as the Circle $ABCD$ is to the Circle $EFGH$, so is the Cone AL to some Solid either less or greater than the Cone EN . First, let it be to the Solid X less than the Cone; and let the Solid I be equal to the Excess of the Cone EN above the Solid X : Then the Cone EN is equal to the Solids X , I . Let the Square $EFGH$ be describ'd in the Circle $EFGH$, which Square is greater than one half of the Circle, and erect a Pyramid upon the Square $EFGH$, of the same Altitude as the Cone; therefore the Pyramid erected is greater than one half of the Cone. For if we describe a Square about the Circle, and a Pyramid be erected thereon, of the same Altitude as the Cone; the Pyramid inscrib'd will be one half of the Pyramid circumscrib'd, for they are * to one another as their Bases; and the Cone is less than the circumscrib'd Pyramid: Therefore the Pyramid whose Base is the Square $EFGH$, and Vertex the same as that of the Cone, is greater than one half of the Cone. Bisect the Circumferences EF , FG , GH , HE , in the Points P , R , S , O , and join HO , OE , EP , PF , FR , RG , GS , SH ; then each of the Triangles HOE , EPF , FRG , GHS , is greater than one half of the Segment of the Circle wherein it is. Let a Pyramid be rais'd upon every one of the Triangles HOE , EPF , FRG , GHS , of the same Altitude as the Cone; then each of those erected Pyramids is greater than one half of its correspondent Segment of the Cone: And so bisecting the remaining Circumferences, joining the right Lines, and erecting Pyramids upon each of the Triangles of

* 6 of this.

the same Altitude as that of the Cone; and doing this continually, there will at last be left Segments of the Cone that will together be less than the Solid I. Let those be the Segments that are on HO , OE , EP , PF , FR , RG , GS , SH ; therefore the Pyramid remaining, whose Base is the Polygon $HOEPFRGS$, and Altitude the same as that of the Cone, is greater than the Solid X. Let the Polygon $DTAYBQCV$ be describ'd in the Circle $ABCD$, similar and alike situate to the Polygon $HOEPFRGS$; and let a Pyramid be erected thereon of the same Altitude as the Cone AL : Then, because the Square of AC to the Square of EG , is * as the Polygon $DTAYBQCV$ * 1 of this, to the Polygon $HOEPFRGS$; and the Square of AC is † to the Square of EG , as the Circle $ABCD$ † 2 of this, to the Circle $EFGH$; it shall be, as the Circle $ABCD$ to the Circle $EFGH$, so is the Polygon $DTAYBQCV$ to the Polygon $HOEPFRGS$. But as the Circle $ABCD$ is to the Circle $EFGH$, so is the Cone AL to the Solid X: And as the Polygon $DTAYBQCV$ is to the Polygon $HOEPFRGS$, so is the Pyramid whose Base is the Polygon $DTAYBQCV$, and Vertex the Point L , to the Pyramid whose Base is the Polygon $HOEPFRGS$, and Vertex the Point N . Therefore, as the Cone AL to the Solid X, so the Pyramid, whose Base is the Polygon $DTAYBQCV$, and Vertex the Point L , to the Pyramid whose Base is the Polygon $HOEPFRGS$, and Vertex the Point N . But the Cone AL is greater than the Pyramid that is in it; therefore the Solid X is greater than the Pyramid that is in the Cone EN . But it was put less, which is absurd; therefore the Circle $ABCD$ to the Circle $EFGH$, is not as the Cone AL to some Solid less than the Cone EN . In like manner it is demonstrated, that the Circle $EFGH$ to the Circle $ABCD$, is not as the Cone EN to some Solid less than the Cone AL : I say, moreover, that the Circle $ABCD$ to the Circle $EFGH$, is not as the Cone AL to some Solid greater than the Cone EN . For, if it be possible, let it be to the Solid Z , greater than the Cone; then, (by Inversion) as the Circle $EFGH$ is to the Circle $ABCD$, so shall the Solid Z be to the Cone AL . But since the Solid Z is greater than the Cone EN , it shall be, as the Solid

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Z

Z is to the Cone AL , so is the Cone EN to some Solid less than the Cone AL ; and therefore, as the Circle $EFGH$ is to the Circle $ABCD$, so is the Cone EN to some Solid less than the Cone AL , which has been prov'd to be impossible. Therefore the Circle $ABCD$ to the Circle $EFGH$, is not as the Cone AL to some Solid greater than the Cone EN . It has also been prov'd, that the Circle $ABCD$ to the Circle $EFGH$, is not as the Cone AL to some Solid less than the Cone EN ; therefore, as the Circle $ABCD$ is to the Circle $EFGH$, so is the Cone AL to the Cone EN : But as Cone is to Cone, so is * Cylinder to Cylinder; for each Cylinder is triple of each Cone; and therefore, as the Circle $ABCD$ is to the Circle $EFGH$, so are Cylinders and Cones standing on them, of the same Altitude. Wherefore *Cones and Cylinders, of the same Altitude, are to one another as their Bases*; which was to be demonstrated.

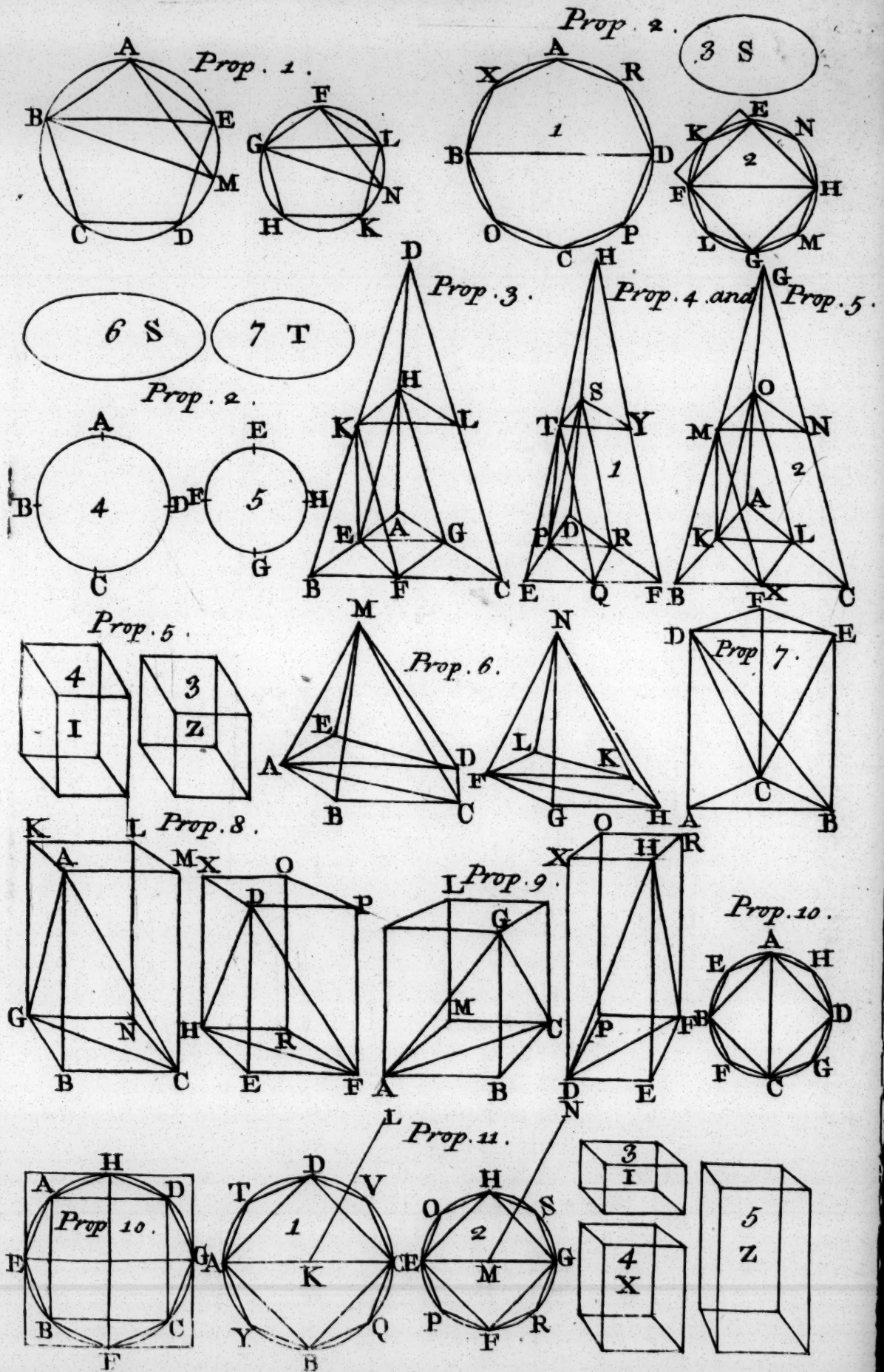
PROPOSITION XII.

THEOREM.

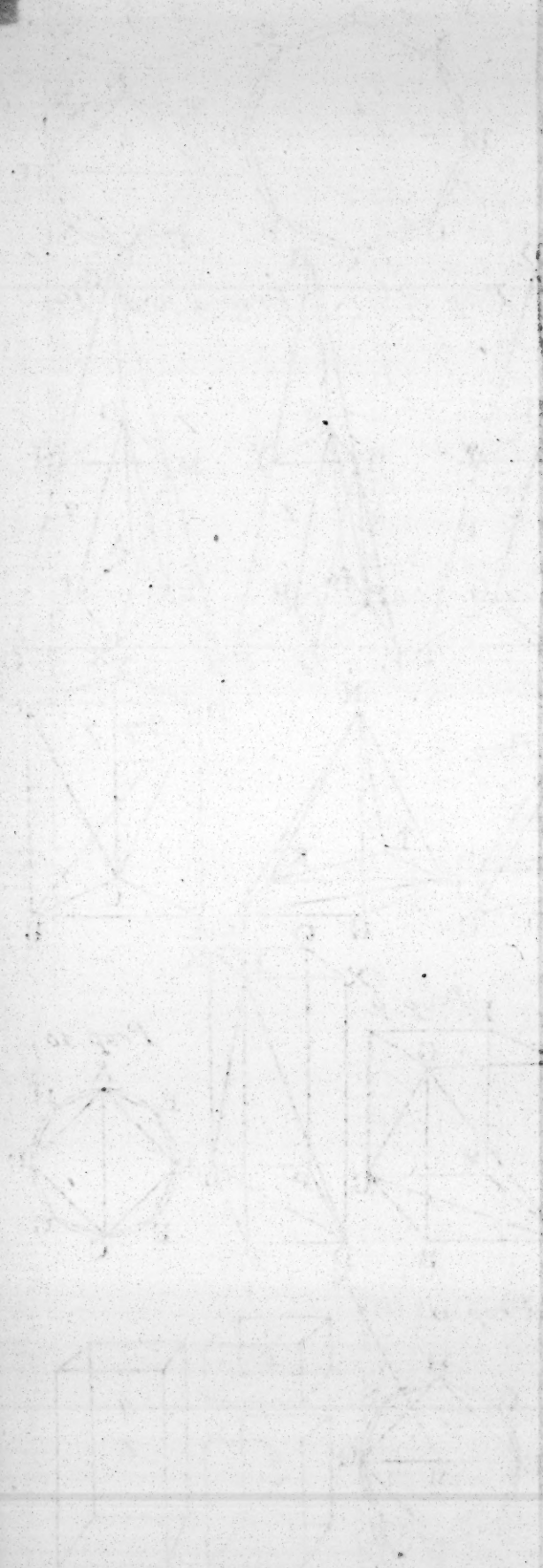
Similar Cones and Cylinders are to one another in a triplicate Proportion of the Diameters of their Bases.

LET there be similar Cones and Cylinders, whose Bases are the Circles $ABCD$, $EFGH$, and Diameters of the Bases BD , FH , and Axes of the Cones or Cylinders KL , MN . I say, the Cone whose Base is the Circle $ABCD$, and Vertex the Point L , to the Cone whose Base is the Circle $EFGH$, and Vertex the Point N , hath a triplicate Proportion of that which BD has to FH .

For, if the Cone $ABCDL$ to the Cone $EFGHN$, has not a triplicate Proportion of that which BD has to FH ; the Cone $ABCDL$ shall have that triplicate Proportion to some Solid, either less or greater than the Cone $EFGHN$. First, let it have that triplicate Proportion to the Solid X , less than the Cone $EFGHN$; and let the Square $EFGH$ be describ'd in the Circle $EFGH$, which will be greater than one half of the Circle $EFGH$; and erect a Pyramid on the



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the Square $EFGH$ of the same Altitude with the Cone, then that Pyramid is greater than one half of the Cone. And so let the Circumferences EF , FG , GH , HE , be bisected in the Points O , P , R , S ; and join EO , OF , FP , PG , GR , RH , HS , SE ; then each of the Triangles EOF , FPG , GRH , HSE , is greater than one half of the Segment of the Circle $EFGH$, in which it is; and erect a Pyramid upon each of the Triangles EOF , EPG , GRH , HSE , having the same Altitude as the Cone: Then each of the Pyramids, thus erected, is greater than half its corresponding Segment of the Cone; wherefore, bisecting the remaining Circumferences, joining right Lines, and erecting Pyramids upon each of the Triangles, having the same Vertex as the Cone; and doing this continually, we shall leave at last certain Segments of the Cone, that shall be less than the Excess by which the Cone $EFGHN$ exceeds the Solid X . Let these be the Segments that stand on EO , OF , FP , PG , GR , RH , HS , SE ; then the remaining Pyramid, whose Base is the Polygon $EOFPGRHS$, and Vertex the Point N , is greater than the Solid X : Also, let the Polygon $ATBYCVDQ$ be describ'd in the Circle $ABCD$, similar and alike situate to the Polygon $EOFPGRHS$; upon which erect a Pyramid having the same Altitude as the Cone; and let LBT be one of the Triangles containing the Pyramid, whose Base is the Polygon $ATBYCVDQ$, and Vertex the Point L ; as, likewise, NFO one of the Triangles containing the Pyramid $EOFPGRHS$, and Vertex the Point N ; and let KT , MO , be join'd. Then, because the Cone $ABCDL$ is similar to the Cone $EFGHN$, it shall be, as BD is to FH , so is the Axis KL to the Axis MN : But as BD is to FH , so is * BK to FM ; and as BK is to FM , conse- * 15. 5. quently so is KL to MN ; and (by Alternation) as BK is to KL , so is FM to MN . And since each is perpendicular, and the Sides about the equal Angles BKL , FMN , are proportional; the Triangle BKL shall be † similar to the Triangle FMN . Again, be- † 6. 6. cause BK is to KT , as FM is to MO , the Sides are proportional about equal Angles BKT , FMO ; for the Angle BKT is the same Part of the four right Angles at the Centre K , as the Angle FMO is of the

* 6. 6.

four right Angles at the Centre M ; the Triangle BKT shall be * similar to the Triangle FMO. And because it has been prov'd that BK is to KL, as FM is to MN ; and BK is equal to KT ; and FM to MO ; it shall be, as TK is to KL, so is OM to MN ; and the proportional Sides are about equal Angles TKL, OMN, for they are right Angles : Therefore the Triangle LKT shall be similar to the Triangle MNO. And since, by the Similarity of the Triangles BKL, FMN, it is, as LB is to BK, so is NF to FM ; and, by the Similarity of the Triangles BKT, FMO, it is, as KB is to BT, so is MF to FO ; it shall be, (by Equality of Proportion) as LB is to BT, so is NF to FO. Again, since, by the Similarity of the Triangles LTK, NOM, it is, as LT is to TK, so is NO to OM ; and, by the Similarity of the Triangles KBT, OMF, it is, as KT is to TB, so is MO to OF ; it shall be, (by Equality of Proportion) as LT is to TB, so is NO to OF. But it has been prov'd, that TB is to BL, as OF is to FN ; wherefore, again, (by Equality of Proportion) as TL is to LB, so is ON to NF ; and therefore the Sides of the Triangles LTB, NOF, are proportional ; and so the Triangles LTB, NOF, are equiangular and similar to each other ; and, consequently, the Pyramid, whose Base is the Triangle BKT, and Vertex the Point L, is similar to the Pyramid whose Base is the Triangle FMO, and Vertex the Point N ; for they are contain'd under similar Planes equal in Multitude : But

† 8 of this. similar Pyramids, that have triangular Bases, are † to one another in the triplicate Proportion of their homologous Sides ; therefore the Pyramid BKT L to the Pyramid FMON, has a triplicate Proportion of that which BK has to FM. In like manner, drawing right Lines from the Points A, Q, D, V, C, Y, to K, as also others, from the Points E, S, H, R, G, P, to M, and erecting Pyramids on the Triangles having the same Vertices as the Cones, we demonstrate, that every Pyramid of one Cone, to every one of the other Cone, has a triplicate Proportion of that which the Side BK has to the homologous Side MF, that is, which BD has to FH. But as one of the Antecedents is to one of the Consequents, so are † all the Antecedents to all the Consequents. Therefore, as

† 12. 5.

the Pyramid $BKTL$ is to the Pyramid $FMON$, so is the whole Pyramid whose Base is the Polygon $ATBYCVDQ$, and Vertex the Point L , to the whole Pyramid, whose Base is the Polygon $EOFPGRHS$, and Vertex the Point N . Wherefore the Pyramid, whose Base is the Polygon $ATBYCVDQ$, and Vertex the Point L , to the Pyramid whose Base is the Polygon $EOFPGRHS$, and Vertex the Point N , has a triplicate Proportion of that which BD hath to FH . But the Cone whose Base is the Circle $ABCD$, and Vertex the Point L , is suppos'd to have to the Solid X a triplicate Proportion of that which BD has to FH ; therefore, as the Cone whose Base is the Circle $ABCD$, and Vertex the Point L , is to the Solid X , so is the Pyramid whose Base is the Polygon $ATBYCVDQ$, and Vertex the Point L , to the Pyramid whose Base is the Polygon $EOFPGRHS$, and Vertex the Point N . But the said Cone is greater than the Pyramid that is in it, for it comprehends it; therefore the Solid X , also, is greater than the Pyramid, whose Base is the Polygon $EOFPGRHS$, and Vertex the Point N : But it is also less, which is absurd; therefore the Cone, whose Base is the Circle $ABCD$, and Vertex the Point L , to some Solid less than the Cone, whose Base is the Circle $EFGH$, and Vertex the Point N , has not a triplicate Proportion of that which BD has to FH . In like manner we demonstrate, that the Cone $EFGHN$, to some Solid less than the Cone $ABCDL$, has not a triplicate Proportion of that which FH has to BD . Lastly, I say, the Cone $ABCDL$, to a Solid greater than the Cone $EFGHN$, has not a triplicate Proportion of that which BD has to FH : For, if this be possible, let it be so to some Solid Z greater than the Cone $EFGHN$; then (by Inversion) the Solid Z , to the Cone $ABCDL$, has a triplicate Proportion of that which FH has to BD . But since the Solid Z is greater than the Cone $EFGHN$, the Solid Z shall be to the Cone $ABCDL$, as the Cone $EFGHN$ is to some Solid less than the Cone $ABCDL$; and therefore the Cone $EFGHN$, to some Solid less than the Cone $ABCDL$, hath a triplicate Proportion of that which FH has to BD , which has been prov'd to be impossible; therefore the Cone $ABCDL$, to

some Solid greater than the Cone $EFGHN$, has not a triplicate Proportion of that which BD has to FH . It has been also demonstrated, that the Cone $AECDL$, to some Solid less than the Cone $EFGHN$, hath not a triplicate Proportion of that which BD has to FH ; wherefore the Cone $AECDL$, to the Cone $EFGHN$, has a triplicate Proportion of that which BD has to FH . But as Cone is to Cone, so is * Cylinder to Cylinder; for a Cylinder, having the same Base as a Cone, and the same Altitude, is † triple of the Cone; since it is demonstrated, that every Cone is one third Part of a Cylinder, having the same Base, and equal Altitude: Therefore, also, a Cylinder to a Cylinder has a triplicate Proportion of that which BD has to FH . Therefore *similar Cones and Cylinders are to one another in a triplicate Proportion of the Diameters of their Bases*; which was to be demonstrated.

* 15. 5.

† 10 of this.

PROPOSITION XIII.

THEOREM.

If a Cylinder be divided by a Plane parallel to the opposite Planes; then, as one Cylinder is to the other Cylinder, so is the Axis to the Axis.

LET the Cylinder AD be divided by the Plane GH , parallel to the opposite Planes AB , CD , and meeting the Axis EF in the Point K . I say, as the Cylinder BG is to the Cylinder GD , so is the Axis EK to the Axis KF .

For, let the Axis EF be both ways produced to L and M ; and put any Number of EN , NL , &c. each equal to the Axis EK ; and any Number of FX , XM , &c. each equal to FK ; and thro' the Points L , N , X , M , let Planes parallel to AB , CD , pass; and in those Planes from L , N , X , M , as Centres, describe the Circles OP , RS , TY , VQ , each equal to AB , CD ; and conceive the Cylinders PR , RB , DT , TQ , to be completed: Then, because the Axes LN , NE , EK , are equal to each other, the Cylinders PR , RB , BG , will be * to one another as their Bases; and therefore the Cylinders PR , RB , BG , are equal: And since the Axes LN , NE , EK , are

* 11 of this.

are equal to each other, as also the Cylinders PR, RB, BG; and the Number of LN, NE, EK, is equal to the Number of PR, RB, BG: The Axis KL shall be the same Multiple of the Axis EK, as the Cylinder PG is of the Cylinder GB. For the same Reason, the Axis MK is the same Multiple of the Axis KF, as the Cylinder GQ is of the Cylinder GD. Now, if the Axis KL be equal to the Axis KM, the Cylinder PG shall be equal to the Cylinder GQ; if the Axis LK be greater than the Axis KM, the Cylinder PG shall be likewise greater than the Cylinder GQ; and, if less, less: Therefore, because there are four Magnitudes, *viz.* the Axes EK, KF, and the Cylinders BG, GD; and there are taken their Equimultiples, namely, the Axis KL, and the Cylinder PG, the Equimultiples of the Axis EK, and the Cylinder BG; and the Axis KM, and the Cylinder GQ, the Equimultiples of the Axis KF, and the Cylinder GD: And it is demonstrated, that if the Axis LK exceeds the Axis KM, the Cylinder PG will exceed the Cylinder GQ; and, if it be equal, equal; and less, less. Therefore, as the Axis EK is to the Axis KF, so * is the Cylinder BG to the Cylinder GD. *Def. 5. 50* Wherefore if a Cylinder be divided by a Plane parallel to the opposite Planes; then, as one Cylinder is to the other Cylinder, so is the Axis to the Axis; which was to be demonstrated.

PROPOSITION XIV.

THEOREM.

Cones and Cylinders, being upon equal Bases, are to one another as their Altitudes.

LET the Cylinders EB, FD, stand upon equal Bases AB, CD. I say, as the Cylinder EB is to the Cylinder FD, so is the Axis GH to the Axis KL.

For, produce the Axis KL to the Point N; and put LN equal to the Axis GH; and let a Cylinder CM be conceiv'd about the Axis LN: Then, because the Cylinders EB, CM, have the same Altitude, they are * to one another as their Bases. But * *11 of this.*

their Bases are equal; therefore the Cylinders EB , CM , will be also equal. And because the Cylinder FM is cut by a Plane CD , parallel to the opposite Planes, it shall be, as the Cylinder CM is to the Cylinder FD , so is the Axis LN to the Axis KL . But the Cylinder CM is equal to the Cylinder EB ; and the Axis LN to the Axis GH ; therefore the Cylinder EB is to the Cylinder FD , as the Axis GH is to the Axis KL : And as the Cylinder EB is to the Cylinder FD , so is $\dagger$ the Cone ABG to the Cone CDK , for the Cylinders are $*$ triple of the Cones. Therefore, as the Axis GH is to the Axis KL , so is the Cone ABG to the Cone CDK ; and so the Cylinder EB to the Cylinder FD . Wherefore *Cones and Cylinders, being upon equal Bases, are to one another as their Altitudes*; which was to be demonstrated.

$\dagger$ 15. 5.

$*$ 10 of this.

PROPOSITION XV.

THEOREM.

The Bases and Altitudes of equal Cones and Cylinders are reciprocally proportional; and Cones and Cylinders, whose Bases and Altitudes are reciprocally proportional, are equal to one another.

LET the Bases of the equal Cones and Cylinders be the Circles $ABCD$, $EFGH$, and their Diameters AC , EG ; and Axes KL , MN ; which are also the Altitudes of the Cones and Cylinders: And let the Cylinders AX , EO , be completed. I say, the Bases and Altitudes of the Cylinders AX , EO , are reciprocally proportional; that is, the Base $ABCD$ is to the Base $EFGH$, as the Altitude MN is to the Altitude KL .

For the Altitude KL is either equal to the Altitude MN , or not equal. First, let it be equal; and the Cylinder AX is equal to the Cylinder EO . But Cylinders and Cones that have the same Altitude, are $*$ to one another as their Bases; therefore the Base $ABCD$ is equal to the Base $EFGH$: And, consequently, as the Base $ABCD$ is to the Base $EFGH$, so is the Altitude MN to the Altitude KL .

$*$ 11 of this.

But

But if the Altitude KL be not equal to the Altitude MN , let MN be the greater; and take PM , equal to LK , from MN ; and let the Cylinder EO be cut thro' P by the Plane $TY S$, parallel to the opposite Planes of the Circles $EFGH$, RO ; and conceive ES to be a Cylinder, whose Base is the Circle $EFGH$, and Altitude PM : Then, because the Cylinder AX is equal to the Cylinder EO , and ES is some other Cylinder; the Cylinder AX to the Cylinder ES , shall be as the Cylinder EO is to the Cylinder ES . But as the Cylinder AX is to the Cylinder ES , so is * the Base $ABCD$ to the Base * 11 of this. $EFGH$; for the Cylinders AX , ES , have the same Altitude: And as the Cylinder EO is to the Cylinder ES , so is † the Altitude MN to the Altitude MP ; † 13 of this. for the Cylinder EO is cut by the Plane $TY S$ parallel to the opposite Planes. Therefore, as the Base $ABCD$ is to the Base $EFGH$, so is the Altitude MN to the Altitude MP . But the Altitude MP is equal to the Altitude KL ; wherefore, as the Base $ABCD$ is to the Base $EFGH$, so is the Altitude MN to the Altitude KL ; and therefore the Bases and Altitudes of the equal Cylinders AX , EO , are reciprocally proportional.

And if the Bases and Altitudes of the Cylinders AX , EO , are reciprocally proportional, that is, if the Base $ABCD$ be to the Base $EFGH$, as the Altitude MN is to the Altitude KL ; I say, the Cylinder AX is equal to the Cylinder EO . For, the same Construction remaining, because the Base $ABCD$ is to the Base $EFGH$, as the Altitude MN is to the Altitude KL ; and the Altitude KL is equal to the Altitude MP ; it shall be, as the Base $ABCD$ is to the Base $EFGH$, so is the Altitude MN to the Altitude MP . But as the Base $ABCD$ is to the Base $EFGH$, so is the Cylinder AX to the Cylinder ES , for they have the same Altitude; and as the Altitude MN is to the Altitude MP , so is ‡ the Cylinder ‡ 11 of this. EO to the Cylinder ES . Therefore, as the Cylinder AX is to the Cylinder ES , so is the Cylinder EO to the Cylinder ES : Wherefore the Cylinder AX is equal to the Cylinder EO . In like manner we prove this in Cones; *which was to be demonstrated.*

PROPOSITION XVI.

PROBLEM.

Two Circles being about the same Centre, to inscribe in the greater a Polygon of equal Sides, even in Number, that shall not touch the lesser Circle.

LET ABCD, EFGH, be two given Circles about the Centre K. It is requir'd to inscribe a Polygon of equal Sides, even in Number, in the Circle ABCD, not touching the lesser Circle EFGH.

Draw the right Line BD through the Centre K, as also AG from the Point G, at right Angles to BD, which produce to C; this Line will * touch the Circle EFGH: Then, bisecting the Circumference BAD, and again bisecting the Half thereof, and doing this continually, we shall have a Circumference left, at last, less than AD. Let this Circumference be LD, and draw LM, from the Point L, perpendicular to BD, which produce to N; and join LD, DN: And then LD is † equal to DN. And since LN is parallel to AC, and AC touches the Circle EFGH, LN will not touch the Circle EFGH; and much less do the right Lines LD, DN, touch the Circle. And if right Lines, each equal to LD, be applied round the Circle ABCD, we shall have a Polygon inscrib'd therein of equal Sides, even in Number, that does not touch the lesser Circle EFG; which was to be demonstrated.

PROPOSITION XVII.

PROBLEM.

To describe a solid Polyhedron, in the greater of two Spheres, having the same Centre, which shall not touch the Superficies of the lesser Sphere.

LET two Spheres be suppos'd about the same Centre A. It is requir'd to describe a solid Polyhedron in the greater Sphere, not touching the Superficies of the lesser Sphere. Let

Let the Spheres be cut by some Plane passing thro' the Centre ; then the Sections will be Circles : For, because a Sphere is * made by the turning of a Semi-^{Def. 14.} circle about the Diameter, which is at Rest ; in what-^{11.} soever Position the Semicircle is conceiv'd to be, the Plane in which it is shall make a Circle in the Superficies of the Sphere. It is also manifest, that this Circle is a great Circle, since the Diameter of the Sphere, which is likewise the Diameter of the Semicircle, is † greater than all right Lines that are drawn in the † 15. 3. Circle or Sphere. Now, let $BCDE$ be that Circle of the greater Sphere, and FGH of the lesser Sphere ; and let BD , CE , be two of their Diameters drawn at right Angles to one another ; let BD meet the lesser Circle in the Point G , from which, to A , let GL be drawn at right Angles, and AL join'd : Then, bisecting the Circumference EB , as also the Half thereof, and doing thus continually, we shall have left, at last, a certain Circumference less than that Part of the Circumference of the Circle BCD , which is subtended by a right Line equal to GL . Let this be the Circumference BK ; then the right Line BK is less than GL ; and BK shall be the Side of a Polygon of equal Sides, even in Number, not touching the lesser Circle : Now, let the Sides of the Polygon, in the Quadrant of the Circle BE , be the right Lines BK , KL , LM , ME ; and produce the Line joining the Points K , A , to N ; and raise † AX † 12. 11. from the Point A , perpendicular to the Plane of the Circle $BCDE$, meeting the Superficies of the Sphere in the Point X ; and let Planes be drawn thro' AX , and BD , and thro' AX , and KN , which, from what has been said, will make great Circles in the Superficies of the Sphere ; and let BXD , KXN , be Semicircles on the Diameters BD , KN : Then, because XA is perpendicular to the Plane of the Circle $BCDE$, all Planes that pass through XA shall also * be perpendicular to * 18. 11. that same Plane. Therefore the Semicircles BXD , KXN , are perpendicular to that same Plane. And because the Semicircles BED , BXD , KXN , are equal, for they stand upon equal Diameters BD , KN ; their Quadrants BE , BX , KX , shall be also equal. And therefore, as many Sides as the Polygon in the Quadrant BE has, so many Sides may there be

be in the Quadrants BX , KX , equal to the Sides BK , KL , LM , ME . Let those Sides be BO , OP , PR , RK , KS , ST , TY , YX ; and join SO , TP , YR ; and let Perpendiculars be drawn from O , S , to the Plane of the Circle $BCDE$: These will † fall on BD , KN , the common Sections of the Planes; because the Planes of the Semicircles BXD , KXN , are perpendicular to the Plane of the Circle $BCDE$: Let the said Perpendiculars be OV , SQ , and join VQ ; then, since the equal Circumferences BO , SK , are taken in the equal Semicircles BXD , KXN , and OV , SQ , are Perpendiculars, OV shall be equal to SQ , and BV to KQ . But the Whole BA is equal to the Whole KA ; therefore the Part remaining VA is equal to the Part remaining QA : Therefore, as BV is to VA , so is KQ to QA ; and so VQ is ‡ parallel to BK . And since OV and SQ are both perpendicular to the Plane of the Circle $BCDE$, OV shall be * parallel to SQ . But it has also been prov'd equal to it; wherefore QV , SO , are ‡ equal and parallel. And because QV is parallel to SO , and also parallel to KB , SO shall be also ‡ parallel to KB : But BO , KS , join them; therefore $KBOS$ is * a quadrilateral Figure in one Plane: For if two right Lines be parallel, and Points be taken in both of them, a right Line joining the said Points is in the same Plane as the Parallels are. And, for the same Reason, each of the quadrilateral Figures $SOPT$, $TPRY$, are in one Plane. And the Triangle YRX is † in one Plane; therefore, if right Lines be suppos'd to be drawn from the Points O , S , P , T , R , Y , to the Point A , there will be constituted a certain solid polyhedrous Figure within the Circumferences BX , KX , compos'd of Pyramids whose Bases are the quadrilateral Figures $KBOS$, $SOPT$, $TPRY$, and the Triangle YRX ; and Vertices the Point A . And if there be made the same Construction on each of the Sides KL , LM , ME , like as we have done on the Side KB , and also in the other three Quadrants, and the other Hemisphere, there will be constituted a polyhedrous Figure describ'd in the Sphere, compos'd of Pyramids whose Bases are the aforesaid quadrilateral Figures; and the Triangle YRX , being of the same Order
and

† 38. 11.

† 2. 6.

* 6. 11.

† 33. 1.

† 9. 11.

* 7. 11.

† 2. 11.

and Vertices as the Point A; I say, the said Polyhedron does not touch the Superficies of the Sphere, wherein the Circle F G H is. Let A Z be drawn † † 11. 11. from the Point A, perpendicular to the Plane of the quadrilateral Figure K B S O, meeting it in the Point Z; and join B Z, Z K: Then, since A Z is perpendicular to the Plane of the quadrilateral Figure K B S O, it shall also be * perpendicular to all right Lines that * Def. 3. 11. touch it, and are in the same Plane: Wherefore A Z is perpendicular to B Z and Z K. And because A B is equal to A K, the Square of A B shall be also equal to the Square of A K; and the Squares of A Z, Z B, are † equal to the Square of A B; for the Angle at † 47. 1. Z is a right Angle; and the Squares of A Z, Z K, are equal to the Square of A K: Therefore the Squares of A Z, Z B, are equal to the Squares of A Z, Z K. Let the common Square of A Z be taken away, and then the Square of B Z, remaining, is equal to the Square of Z K remaining, and so the right Line B Z is equal to the right Line Z K. After the same manner we demonstrate, that right Lines drawn from the Point Z to the Points O, S, are each equal to B Z, Z K. Therefore a Circle describ'd about the Centre Z, with either of the Distances Z B, Z K, will also pass thro' the Points O, S. And because B K S O is a quadrilateral Figure in a Circle, and O B, B K, K S, are equal, and O S is less than B K; the Angle B Z K shall be obtuse; and so B K greater than B Z. But G L, also, is much greater than B K; therefore G L is greater than B Z; and the Square of G L is greater than the Square of B Z. And since A L is equal to A B, the Square of A L shall be equal to the Square of A B. But the Squares of A G, G L, together, are equal to the Square of A L; and the Squares of B Z, Z A, together, equal to the Square of A B. Therefore the Squares of A G, G L, together, are equal to the Squares of B Z, Z A, together. But the Square of B Z is less than the Square of G L; therefore the Square of Z A is greater than the Square of A G; and so the right Line Z A will be greater than the right Line A G. But A Z is perpendicular to one Base of the Polyhedron, and A G to the Superficies; wherefore the Polyhedron does not touch the Superficies of the lesser Sphere. There-
fore

fore there is describ'd a solid Polyhedron in the greater of two Spheres, having the same Centre, which doth not touch the Superficies of the lesser Sphere ; which was to be demonstrated.

Coroll. Also, if a solid Polyhedron be describ'd in some other Sphere, similar to that which is describ'd in the Sphere BCDE ; the solid Polyhedron describ'd in the Sphere BCDE, to the solid Polyhedron describ'd in that other Sphere, shall have a triplicate Proportion of that which the Diameter of the Sphere BCDE hath to the Diameter of that other Sphere. For the Solids being divided into Pyramids, equal in Number, and of the same Order, the same Pyramids shall be similar. But similar Pyramids are to each other in a triplicate Proportion of their homologous Sides ; therefore the Pyramid whose Base is the quadrilateral Figure KBOS, and Vertex the Point A, to the Pyramid of the same Order into the other Sphere, has a triplicate Proportion of that which the homologous Side of one has to the homologous Side of the other ; that is, which AB, drawn from the Centre A of the Sphere, to that Line which is drawn from the Centre of the other Sphere. In like manner, every one of the Pyramids, that are in the Sphere whose Centre is A, to every one of the Pyramids of the same Order in the other Sphere, hath a triplicate Proportion of that which AB has to that Line drawn from the Centre of the other Sphere: And as one of the Antecedents is to one of the Consequents, so are all the Antecedents to all the Consequents. Wherefore the whole solid Polyhedron, which is in the Sphere describ'd about the Centre A, to the whole solid Polyhedron that is in the other Sphere, hath a triplicate Proportion of that which AB hath to the Line drawn from the Centre of the other Sphere ; that is, which the Diameter BD has to the Diameter of the other Sphere.

PROPOSITION XVIII.

THEOREM.

Spheres are to one another in a triplicate Proportion of their Diameters.

SUPPOSE ABC , DEF , are two Spheres, whose Diameters are BC , EF . I say, the Sphere ABC to the Sphere DEF , has a triplicate Proportion of that which BC has to EF .

For, if it be not so, the Sphere ABC to a Sphere either lesser or greater than DEF , will have a triplicate Proportion of that which BC has to EF . First, let it be to a lesser, as GHK ; and suppose the Sphere DEF to be describ'd about the Sphere GHK ; and let there be describ'd * a solid Polyhedron in the greater Sphere DEF , not touching the Superficies of the lesser Sphere GHK ; also, let a solid Polyhedron be describ'd in the Sphere ABC , similar to that which is describ'd in the Sphere DEF ; then the solid Polyhedron in the Sphere ABC , to the solid Polyhedron in the Sphere DEF , will have † a triplicate Proportion of that which BC has to EF : But the Sphere ABC to the Sphere GHK , hath a triplicate Proportion of that which BC hath to EF ; therefore, as the Sphere ABC is to the Sphere GHK , so is the solid Polyhedron in the Sphere ABC to the solid Polyhedron in the Sphere DEF ; and (by Inversion) as the Sphere ABC is to the solid Polyhedron that is in it, so is the Sphere GHK to the solid Polyhedron that is in the Sphere DEF . But the Sphere ABC is greater than the solid Polyhedron that is in it; therefore the Sphere GHK is also greater than the solid Polyhedron that is in the Sphere DEF , and also less than it, as being comprehended thereby, which is absurd; therefore the Sphere ABC to a Sphere less than the Sphere DEF , hath not a triplicate Proportion of that which BC has to EF . After the same manner it is demonstrated, that the Sphere DEF to a Sphere less than ABC , has not a triplicate Proportion of that which EF has to BC : I say, moreover, that the Sphere ABC to a Sphere greater than DEF , hath not a triplicate

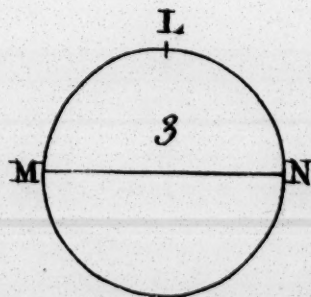
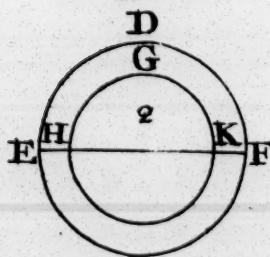
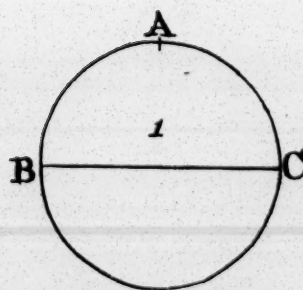
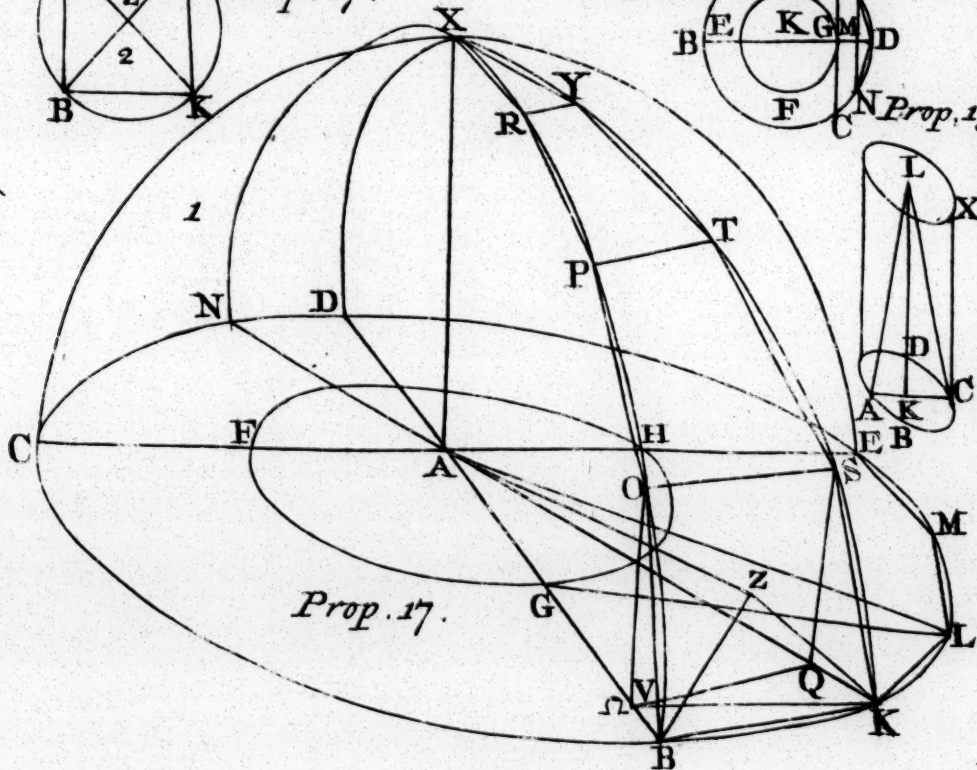
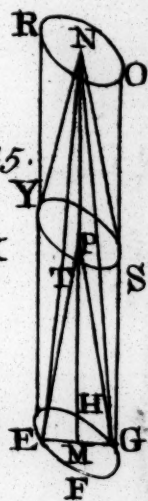
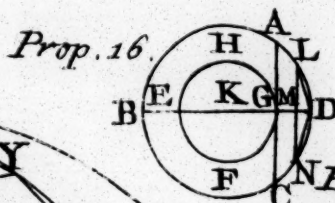
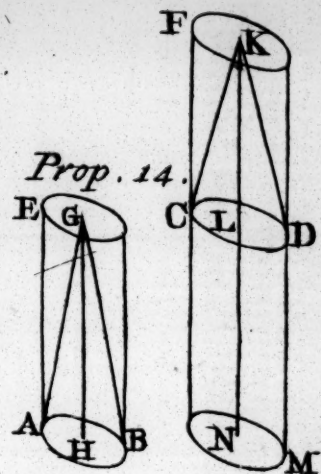
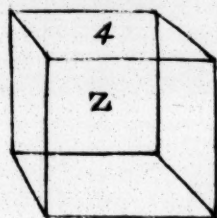
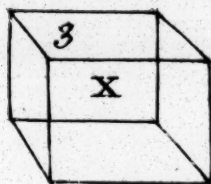
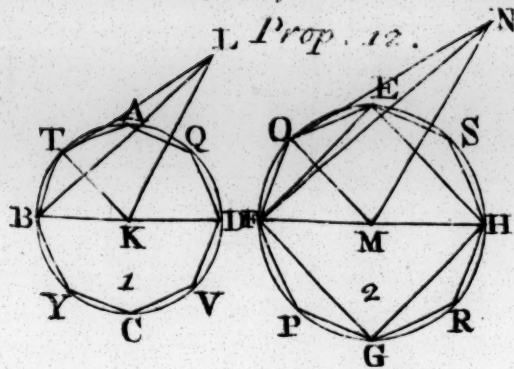
* 17 of this.

† Cor. to the last Prop.

plicate Proportion of that which BC has to EF: For
 if it be possible, let it have to the Sphere LMN greater
 than DEF; then (by Inversion) the Sphere LMN
 to the Sphere ABC, shall have a triplicate Proportion
 of that which the Diameter EF has to the Diameter
 BC. But as the Sphere LMN is to the Sphere ABC,
 so is the Sphere DEF to some Sphere less than ABC,
 because the Sphere LMN is greater than DEF.
 Therefore the Sphere DEF to a Sphere less than
 ABC, hath a triplicate Proportion to that which EF
 has to BC, which is absurd, and has been before
 prov'd. Therefore the Sphere ABC to a Sphere
 greater than DEF, has not a triplicate Proportion of
 that which BC has to EF. But it has also been demon-
 strated, that the Sphere ABC to a Sphere less than
 DEF, has not a triplicate Proportion of that which
 BC has to EF; therefore the Sphere ABC to the
 Sphere DEF, has a triplicate Proportion of that
 which BC has to EF; *which was to be demonstrated.*



F I N I S.



2

T H E
E L E M E N T S
Of Plain and Spherical
T R I G O N O M E T R Y.

D E F I N I T I O N S.

TH E Business of Trigonometry is to find the Angles when the Sides are given, and the Sides, or the Ratios of the Sides, when the Angles are given, and to find Sides and Angles, when Sides and Angles are given: In order to which, it is necessary, that not only the Peripheries of Circles, but also certain right Lines in and about Circles, be suppos'd divided into some determined Number of Parts.

And so the ancient Mathematicians thought fit to divide the Periphery of a Circle into 360 Parts, which they call Degrees; and every Degree into 60 Minutes, and every Minute into 60 Seconds; and again, every Second into 60 Thirds, and so on. And every Angle is said to be of such a Number of Degrees and Minutes, as there are in the Arc measuring that Angle.

There are some that would have a Degree divided into centesimal Parts, rather than sexagesimal ones: And it would perhaps be more useful to divide, not only a Degree, but even the whole Circle, in a decuple Ratio; which Division may some time or other gain Place. Now, if a Circle contains 360 Degrees, a Quadrant thereof, which is the Measure of a right Angle, will
T be

The ELEMENTS of

be 90 of those Parts: And if it contains 100 Parts, a Quadrant will be 25 of these Parts.

The Complement of an Arc is the Difference thereof from a Quadrant.

A Chord, or Subtense, is a right Line drawn from one End of the Arc to the other.

The Right Sine of any Arc, which also is commonly called only a Sine, is a Perpendicular falling from one End of an Arc, to the Radius drawn through the other End of the said Arc; and is therefore the Semisubtense of double the Arc, viz. $DE = \frac{1}{2} DO$, and the Arc DO is double of the Arc DB. Hence, the Sine of an Arc of 30 Degrees is equal to the one half of the Radius. For (by 15 El. 4.) the Side of an Hexagon inscrib'd in a Circle, that is, the Subtense of 60 Degrees, is equal to the Radius. A Sine divides the Radius into two Segments CE, EB; one of which, CE, which is intercepted between the Centre and the Right Sine, is the Sine Complement of the Arc DB to a Quadrant, (for $CE = FD$, which is the Sine of the Arc DH) and is called the Cosine: The other Segment BE, which is intercepted between the Right Sine and the Periphery, is called a Versed Sine, and sometimes a Sagitta.

And if the right Line CG be produced from the Centre C, thro' one End D of the Arc, until it meets the right Line BG, which is perpendicular to the Diameter drawn thro' the other End B of the Arc, then CG is called the Secant, and BG the Tangent of the Arc DB.

The Cosecant and Cotangent of an Arc is the Secant or Tangent of that Arc, which is the Complement of the former Arc to a Quadrant. Note, As the Chord of an Arc, and of its Complement to a Circle, is the same; so, likewise, is the Sine, Tangent, and Secant of an Arc, the same as the Sine, Tangent, and Secant of its Complement to a Semicircle.

The Sinus Totus is the greatest Sine, or the Sine of 90 Degrees, which is equal to the Radius of the Circle.

A Trigonometrical Canon is a Table, which, beginning from one Minute, orderly expresses the Lengths that every Sine, Tangent, and Secant, have, in respect of the Radius, which is supposed Unity; and is conceived to be divided into 10,000,000 or more decimal Parts. And

so the Sine, Tangent, or Secant, of any Arc, may be had by Help of this Table; and, contrariwise, a Sine, Tangent, or Secant, being given, we may find the Arc it expresses. Take notice, That in the following Tract, R. signifies the Radius, S. a Sine, Cos. a Cosine, T. a Tangent, and Cot. a Cotangent.

The CONSTRUCTION of the Trigonometrical Canon.

PROPOSITION I.

THEOREM.

The two Sides of any right-angled Triangle being given, the other Side is also given.

FOR (by 47 of the first Element) $ACq = ABq + BCq$ and $ACq - BCq = ABq$, and interchangeably $ACq - ABq = BCq$. Whence, by the Extraction of the Square Root, there is given $AC = \sqrt{ABq + BCq}$ and $AB = \sqrt{ACq - BCq}$. And $BC = \sqrt{ACq - ABq}$.

PROPOSITION II.

PROBLEM.

The Sine DE of the Arc BD, and the Radius CD, being given, to find the Cosine DF.

THE Radius CD, and the Sine DE, being given in the right-angled Triangle CDE, there will be given (by the last Prop.) $\sqrt{CDq - DEq} = DF$.

PROPOSITION III.

PROBLEM.

The Sine DE of any Arc DB being given, to find DM or BM the Sine of half the Arc.

DE being given, CE (by the last Prop.) will be given, and accordingly EB, which is the Difference between the Cosine and Radius. Therefore DE, EB, being given in the right-angled Triangle DBE, there will be given DB, whose Half DM is the Sine of the Arc $DL = \frac{1}{2}$ the Arc BD.

PROPOSITION IV.

PROBLEM.

The Sine BM of the Arc BL being given, to find the Sine of double that Arc.

THE Sine BM being given, there will be given (by Prop. 2.) the Cosine CM. But the Triangles CBM, DBE, are equiangular, because the Angles at E and M are right Angles, and the Angle at B common: Wherefore (by 4. 6.) we have $CB : CM :: BD$, or $2 BM : DE$. Whence, since the three first Terms of this Analogy are given, the fourth also, which is the Sine of the Arc DB, will be known.

Coroll. Hence, $CB : 2 CM :: BD : 2 DE$; that is, the Radius is to double the Cosine of one Half of the Arc DB, as the Subtense of the Arc DB is to the Subtense of double that Arc. Also, $CB : 2 CM :: (2 BM : 2 DE ::) BM : DE :: \frac{1}{2} CB : CM$. Wherefore the Sine of any Arc, and the Sine of its Double, being given, the Cosine of the Arc itself is given.

PROPOSITION V.

The Sines of two Arcs BD, FD, being given, to find FI the Sine of the Sum, as likewise EL, the Sine of their Difference.

LET the Radius CD be drawn, and then CO is the Cosine of the Arc FD, which accordingly is given, and draw OP thro' O parallel to DK; also, let OM, GE, be drawn parallel to CB: Then, because the Triangles CDK, COP, CHI, FOH, FOM, are equiangular; in the first Place, $CD : DK :: CO : OP$, which, consequently, is known. Also, we have $CD : CK :: FO : FM$, and so likewise this shall be known. But because $FO = EO$, then will $FM = MG = ON$; and so $OP + FM = FI =$ Sine of the Sum of the Arcs: And $OP - FM$; that is, $OP - ON = EL =$ Sine of the Difference of the Arcs. W. W. D.

Coroll. Because the Differences of the Arcs BE, BD, BF, are equal, the Arc BD shall be an Arithmetical Mean between the Arcs BE, BF.

PROPOSITION VI.

The same Things being supposed, Radius is to double the Cosine of the mean Arc, as the Sine of the Difference to the Difference of the Sines of the Extremes.

FOR we have $CD : CK :: FO : FM$: Whence, by doubling the Consequents $CD : 2 CK :: FO : 2 FM$, or to FG; which is the Difference of the Sines EL, FI. W. W. D.

Coroll. If the Arc BD be 60 Degrees, the Difference of the Sines FI, EL, shall be equal to the Sine FO, of the Distance. For, in this Case, CK is the Sine of 30 Degrees; double thereof being equal to Radius: And so, since $CD = 2 CK$, we shall have $FO = FG$. And, consequently, if the two Arcs BE, BF, are equidistant from the Arc of 60

Degrees, the Difference of the Sines shall be equal to the Sine of the Distance F D.

Coroll. 2. Hence, if the Sines of all Arcs be given distant from one another by a given Interval, from the Beginning of a Quadrant to 60 Degrees, the other Sines may be found by one Addition only: For the Sine of 61 Degrees = Sine of 59 Degrees + Sine of 1 Degree; and the Sine of 62 Degrees = Sine of 58 Degrees + Sine of 2 Degrees: Also, the Sine of 63 Degrees = Sine of 57 Degrees + Sine of 3 Degrees, and so on.

Coroll. 3. If the Sines of all Arcs, from the Beginning of a Quadrant to any Part of the Quadrant, distant from each other by a given Interval, be given, thence we may find the Sines of all Arcs to the Double of that Part. For Example, Let all the Sines to 15 Degrees be given; then, by the precedent Analogy, all the Sines to 30 Degrees, may be found: For Radius is to double the Cosine of 15 Degrees, as the Sine of 1 Degree is to the Difference of the Sines of 14 Degrees, and 16 Degrees. So, also, is the Sine of 3 Degrees, to the Difference between the Sines of 12 and 18 Degrees; and so on continually, until you come to the Sine of 30 Degrees.

After the same manner, as Radius is to double the Cosine of 30 Degrees, or to double the Sine of 60 Degrees, so is the Sine of 1 Degree to the Difference of the Sines of 29 and 31 Degrees: Sine 2 Degrees, to the Difference of the Sines of 28 and 32 Degrees: Sine 3 Degrees, to the Difference of the Sines of 27 and 33 Degrees. But in this Case, Radius is to double the Cosine of 30 Degrees, as 1 to $\sqrt{3}$ *. And, accordingly, if the

FIG. for the DEFINITIONS.

* Let B D be an Arch of 30°

Rad. Tan. Co-sine Sine

Then as CB : BG :: FD : DE. DO = CB ergo DE = $\frac{1}{2}$
 $\sqrt{CB^2 - DE^2} = CE = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}}$. CB : CE :: 1 : $\sqrt{\frac{3}{4}}$.
 CD : 2 CE :: 1 : $2\sqrt{\frac{3}{4}} = \sqrt{\frac{3}{1}} = \sqrt{3}$. Q. E. D.

Sines

Sines of the Distances from the Arc of 30 Degrees, be multiplied by $\sqrt{3}$, the Differences of the Sines will be had.

So, likewise, may the Sines of the Minutes in the Beginning of the Quadrant be found, by having the Sines and Cosines of one and two Minutes given. For, as the Radius is to double the Cosine of 2 :: Sine 1 : Difference of the Sines of 1' and 3' :: Sine 2' : Difference of the Sines of 0' and 4', that is, to the Sine of 4'. And so, the Sines of the four first Minutes being given, we may thereby find the Sines of the others to 8, and from thence to 16, and so on.

PROPOSITION VII.

THEOREM.

In small Arcs, the Sines and Tangents of the same Arcs are nearly to one another, in a Ratio of Equality.

FOR, because the Triangles CED, CBG, are equiangular, $CE : CB :: ED : BG$. But as the Point E approaches B, EB will vanish in respect of the Arc BD; whence CE will become nearly equal to CB, and so ED will be also nearly equal to BG. If EB be less than the $\frac{1}{10,000,000}$ Part of the Radius, then the Difference between the Sine and the Tangent will be also less than the $\frac{1}{10,000,000}$ Part of the Tangent.

Coroll. Since any Arc is less than the Tangent, and greater than its Sine, and the Sine and Tangent of a very small Arc are nearly equal; it follows, that the Arc shall be nearly equal to its Sine: And so, in very small Arcs, it shall be, as Arc is to Arc, so is Sine to Sine.

PROPOSITION VIII.

To find the Sine of the Arc of one Minute.

THE Side of a Hexagon inscrib'd in a Circle, that is, the Subtense of 60 Degrees, is equal to the Radius, (*by 15th of the 4th*) and so the Half of the Radius shall be the Sine of the Arc of 30 Degrees. Wherefore the Sine of the Arc of 30 Degrees being given, the Sine of the Arc of 15 Degrees may be found (*by Prop. 3.*) Also, the Sine of the Arc of 15 Degrees being given, (*by the same Prop.*) we may have the Sine of 7 Degrees 30 Minutes: So, likewise, can we find the Sine of the Half of this, *viz.* 3 Degrees 45'; and so on, until twelve Bisections being made, we come to an Arc of $52^2, 44^3, 03^4, 45^5$, whose Cosine is nearly equal to the Radius, in which Case (*as is manifest from Prop. 7.*) Arcs are proportional to their Sines: And so, as the Arc of $52^2, 44^3, 3^4, 45^5$, is to an Arc of one Minute, so shall the Sine before found be to the Sine of an Arc of one Minute, which therefore will be given. And when the Sine of one Minute is found, then (*by Prop. 2. and 4.*) the Sine and Cosine of two Minutes will be had.

PROPOSITION IX.

THEOREM.

If the Angle BAC, being in the Periphery of a Circle, be bisected by the right Line AD, and if AC be produced until DE = AD meets it in E; then shall CE = AB.

IN the quadrilateral Figure ABDC (*by 22. 3.*) the Angles B and ACD are equal to two right Angles = DCE + DCA (*by 13. 1.*): Whence the Angle B = DCE. But, likewise, the Angle E = DAC (*by 5. 1.*) = DAB, and DC = DB; wherefore the Triangles BAD and CED are congruous, and CE is equal to AB. W. W. D.

PROPOSITION X.

THEOREM.

Let the Arcs AB, BC, CD, DE, EF, &c. be equal; and let the Subtenses of the Arcs AB, AC, AD, AE, &c. be drawn; then will $AB : AC :: AC : AB + AD :: AD : AC + AE :: AE : AD + AF :: AF : AE + AG$.

LET AD be produced to H, AE to I, AF to K, and AG to L, that the Triangles ACH, ADI, AEK, AFL, be Iſoſceles ones: Then, becauſe the Angle BAD is biſected, we ſhall have $DH = AB$ (by the laſt Prop.); ſo likewiſe ſhall $EI = AC$, $FK = AD$, alſo $GL = AE$.

But the Iſoſceles Triangles ABC, CHA, DAI, EAK, FAL, becauſe of the equal Angles at the Baſes, are equiangular: Wherefore it ſhall be, as $AB : AC :: AC : AH = AB + AD :: AD : AI = AC + AE :: AE : AK = AD + AF :: AF : AL = AE + AG$. W. W. D.

Coroll. 1. Becauſe AB is to AC, as Radius is to double the Coſine of $\frac{1}{2}$ the Arc AB, it ſhall alſo be (by Coroll. Prop. 4.) as Radius is to double the Coſine of $\frac{1}{2}$ the Arc AB, ſo is $\frac{1}{2} AB : \frac{1}{2} AC :: \frac{1}{2} AC : \frac{1}{2} AB + \frac{1}{2} AD :: \frac{1}{2} AD : \frac{1}{2} AC + \frac{1}{2} AE :: \frac{1}{2} AE + \frac{1}{2} AD + \frac{1}{2} AF$, &c. Now let each of the Arcs AB, BC, CD, &c. be 2'; then will $\frac{1}{2} AB$ be the Sine of one Minute, $\frac{1}{2} AC$ the Sine of 2 Minutes, $\frac{1}{2} AD$ the Sine of 3 Minutes, $\frac{1}{2} AE$ the Sine of 4', &c. Whence, if the Sines of one and two Minutes be given, we may eaſily find all the other Sines in the following manner:

Let the Coſine of the Arc of one Minute, that is, the Sine of the Arc of 89 Deg. 59', be call'd Q, and make the following Analogies; $R : 2Q :: \text{Sin. } 2' : S. 1' + S. 3'$. Wherefore the Sine of 3 Minutes will be given. Alſo, $R : 2Q :: S. 3' : S. 2' + S. 4'$. Wherefore the S. 4' is given; and $R. 2Q :: S. 4' : S. 3' + S. 5'$; and ſo the Sine of 5' will be had.

Like-

Likewise, $R : 2Q :: S. 5' : S. 4' + S. 6'$; and so we shall have the Sine of $6'$. And, in like manner, the Sines of every Minute of the Quadrant will be given. And because the Radius, or the first Term of the Analogy, is Unity, the Operations will be with great Ease and Expedition calculated by Multiplication, and contracted by Addition. When the Sines are found to 60 Degrees, all the other Sines may be had by Addition only (by *Cor. 1. Prop. 6.*).

The Sines being given, the Tangents and Secants may be found from the following Analogies (in the Figure for the Definitions); because the Triangles CED, CBG, CHI, are equiangular, we have

$CE : ED :: CB : BG$; that is, $\text{Cof.} : S :: R : T$.
 $GB : BC :: CH : HI$; that is, $T : R :: R : \text{Cot.}$
 $CE : CD :: CB : CG$; that is, $\text{Cof.} : R :: R : \text{Secant.}$
 $DE : CD :: CH : CI$; that is, $S : R :: R : \text{Cofec.}$

SCHOLIUM.

That great Geometrician, and incomparable Philosopher, Sir Isaac Newton, was the first that laid down a Series converging, in infinitum; from which, having the Arcs given, their Sines may be found. Thus, if an Arc be called A, and the Radius be an Unit, the Sine thereof will be found to be

$$\begin{array}{ccccccc}
 & A^3 & & A^5 & & A^7 & & A^9 & & \text{\&c.} \\
 A - & \text{---} & + & \text{---} & + & \text{---} & + & \text{---} & + & \text{---} \\
 & 1.2.3 & & 1.2.3.4.5 & & 1.2.3.4.5.6.7 & & 1.2.3.4.5.6.7.8.9. & & \\
 \text{And the Cosine,} & & & & & & & & & \\
 & A^2 & & A^4 & & A^6 & & A^8 & & \text{\&c.} \\
 1 - & \text{---} & + & \text{---} & + & \text{---} & + & \text{---} & + & \text{---} \\
 & 1.2.3 & & 1.2.3.4 & & 1.2.3.4.5.6 & & 1.2.3.4.5.6.7.8 & &
 \end{array}$$

These Series in the Beginning of the Quadrant, when the Arc A is but small, soon converge. For in the Series for the Sine, if A does not exceed 10 Minutes, the two first Terms thereof, viz. $A - \frac{1}{6} A^3$ gives the Sine to 15 Places of Figures. If the Arc A be not greater than one Degree, the three first Terms will exhibit the Sine to 15 Places of Figures; and so the said Series are very useful for finding the first and last Sines of the Quadrant. But the greater the Arc A is, the more

are the Terms of the Series requir'd to have the Sine in Numbers true to a given Place of Figures. And then, when the Arc is nearly equal to the Radius, the Series converges very slow. And therefore, to remedy this, I have devis'd other Series, similar to the Newtonian one, wherein, I suppose, the Arc, whose Sine is sought, is the Sum and Difference of two Arcs, viz. $A+z$, or $A-z$: And let the Sine of the Arc A be call'd a , and the Cosine b . Then the Sine of the Arc $A+z$ will be express'd thus:

$$1. a + \frac{bz}{a} - \frac{az^2}{1.2} - \frac{bz^3}{1.2.3} + \frac{az^4}{1.2.3.4} + \frac{bz^5}{1.2.3.4.5} \text{ \&c.}$$

And the Cosine is,

$$2. b - \frac{az}{1} - \frac{bz^2}{1.2} + \frac{az^3}{1.2.3} + \frac{bz^4}{1.2.3.4} - \frac{az^5}{1.2.3.4.5} - \frac{bz^6}{1.2.3.4.5.6} \text{ \&c.}$$

In like manner the Sine of the Arc $A-z$ is,

$$3. a - \frac{bz}{1} - \frac{az^2}{1.2} + \frac{az^3}{1.2.3} + \frac{az^4}{1.2.3.4} - \frac{bz^5}{1.2.3.4.5} - \frac{az^6}{1.2.3.4.5.6} \text{ \&c.}$$

And the Cosine is,

$$4. b + \frac{az}{1} - \frac{bz^2}{1.2} - \frac{az^3}{1.2.3} + \frac{bz^4}{1.2.3.4} + \frac{az^5}{1.2.3.4.5} \text{ \&c.}$$

The Arc A is an Arithmetical Mean between the Arc $A-z$ and $A+z$. And the Difference of the Sines are,

$$5. \frac{bz}{1} - \frac{az^2}{1.2} - \frac{bz^3}{1.2.3} + \frac{az^4}{1.2.3.4} + \frac{bz^5}{1.2.3.4.5} - \frac{az^6}{1.2.3.4.5.6} \text{ \&c.}$$

$$6. \frac{bz}{1} + \frac{az^2}{1.2} - \frac{bz^3}{1.2.3} - \frac{az^4}{1.2.3.4} + \frac{bz^5}{1.2.3.4.5} + \frac{az^6}{1.2.3.4.5.6} \text{ \&c.}$$

Whence the Difference of the Differences or second Difference.

$$7. \text{Produce } \frac{2az^2}{1.2} - \frac{2az^4}{1.2.3.4} + \frac{2az^6}{1.2.3.4.5.6} \text{ \&c.}$$

$$\text{Or } 2a \times \frac{z^2}{1.2} - \frac{z^4}{1.2.3.4} + \frac{z^6}{1.2.3.4.5.6} \text{ \&c.}$$

Which Series is equal to double the Sine of the mean Arc, drawn into the versed Sine of the Arc z , and converges very soon. So that if z be the first Minute of the Quadrant, the first Term of the Series gives the second Difference to 15 Places of Figures, and the second Term to 25 Places.

From

From hence, if the Sines of the Arcs distant one Minute from each other be given, the Sines of all the Arcs that are in the same Progression may be found by an exceeding easy Operation.

In the first and second Series, if $A=0$; then shall $a=0$, and b its Cosine, will become Radius, or 1. And hence, if the Terms wherein a is, are taken away, and 1 to be put instead of b , the Series will become the Newtonian. In the third and fourth Series, if A be 90 Degrees, we shall have $b=0$, and $a=1$. Whence again, taking away all the Terms wherein b is, and putting 1 instead of a , we shall have the Newtonian Series arise.

Note, All the said Series easily flow from the Newtonian ones. By the fifth Proposition.

PROPOSITION XI.

In a right-angled Triangle, if the Hypothenuse be made the Radius, then are the Sides the Sines of their opposite Angles; and if either of the Legs be made the Radius, then the other Leg is the Tangent of its opposite Angle, and the Hypothenuse is the Secant of that Angle.

IT is manifest, that CB is the Sine of the Arc CD , and AB the Cosine thereof; but the Arc CD is the Measure of the Angle A , and the Complement of the Measure of the Angle C : Moreover, if AB , in the Figure to this Proposition, be suppos'd Radius, then BC is the Tangent, and AC the Secant of the Arc BD , which is the Measure of the Angle A . So, also, if BC be made the Radius, then is BA the Tangent, and AC the Secant of the Arc BE , or Angle C . W. W. D. Therefore, as AC , being taken as some given Measure, is to BC taken in the same Measure; so shall the Number 10,000,000 Parts, in which the Radius is suppos'd to be divided, be to a Number expressing in the same Parts the Length of the Sine of the Angle A ; that is, it will be,

as $AC : BC :: R : S, A.$
 by the same Reason, as $AC : BA :: R : S, C.$
 also, as $AB : BC :: R : T, A.$
 and $BC : BA :: R : T, C.$

And so, if any three of these Proportionals be given, the fourth may be found by the *Rule of Three*.

PROPOSITION XII.

The Sides of plain Triangles are as the Sines of their opposite Angles.

IF the Sides of a Triangle, inscrib'd in a Circle, be bisected by perpendicular Radii, then shall the half Sides be the Sines of the Angles at the Periphery; for the Angle BDC at the Centre is double of the Angle BAC at the Periphery (by 20 *El. lib. 3.*); and so the Half of every of them, *viz.* $BDE = BAC$, and BE is the Sine thereof. For the same Reason, BF shall be the Sine of the Angle BCA, and AG the Sine of the Angle ABC.

In a right-angled Triangle we have $BD = \frac{1}{2} BC = \text{Radius}$ (by 31 *Eucl. 3.*); but Radius is the Sine of a right Angle: Whence $\frac{1}{2} BC$ is the Sine of the Angle A.

In an obtuse-angled Triangle, let BI, CI, be drawn; and then the Angle L shall be the Complement of the Angle A to two right Angles, (by 22 *El. 3.*) and so they shall both have the same Sine. But the Angle BDE (whose Sine is BE) = Angle L; therefore BE shall be the Sine of the Angle BAC. And so in every Triangle, the Halves of the Sides are the Sines of the opposite Angles: But it is manifest, that the Sides are to one another as their Halves. W. W. D.

PROPOSITION XIII.

In a plain Triangle, the Sum of the Legs, the Difference of the Legs, the Tangent of the half Sum of the Angles at the Base, and the Tangent of one half their Difference, are proportional.

LET there be a Triangle ABC, whose Legs are AB, BC, and Base AC. Produce AB to H, so that BH = BC; then shall AH be the Sum of the Legs; and if you make BI = BA, then IH will be the Difference of the Legs. Also, the Angle HBC = Angles A + ACB, (by 32 El. 1.) and so EBC the Half thereof = half the Sum of the Angles A and ACB, and its Tangent (putting the Radius = EB) is EC. Again, let BD be drawn parallel to AC, and make HF = CD; then, since HB = CB, we shall have (by 4 El. 1.) the Angle HBF = CBD = BCA (by 29 El. 1.) Also, the Angle HBD = Angle A; whence FBD shall be the Difference of the Angles A and ACB; and EBD, whose Tangent is ED, half their Difference. Let IG be drawn thro' I parallel to AC or BD, and then (by 2 El. 6.) AB : BI :: CD : DG. But AB = BI; whence we shall have CD = DG: But CD = HF, and so HF = DG; and, consequently, HG = DF, and $\frac{1}{2}$ HG = $\frac{1}{2}$ DF = DE; and because the Triangles AHC, IHG, are equiangular, it shall be, as AH : IH :: HC : HG :: $\frac{1}{2}$ HC : $\frac{1}{2}$ HG :: EC : ED. That is, AH the Sum of the Legs, to IH the Difference of the Legs, shall be, as EC the Tangent of one half the Sum of the Angles at the Base, to ED the Tangent of one half their Difference. W. W. D.

PROPOSITION XIV.

In a plain Triangle, the Base, the Sum of the Sides, the Difference of the Sides, and the Difference of the Segments of the Base, are proportional.

LET DC be the Base of the Triangle BCD. About the Centre B, with the Radius BC, let a Circle be describ'd; produce DB to G, and from B
let

Let fall BE perpendicular to the Base; then shall $DG = DB + BC =$ Sum of the Sides, and $DH =$ Difference of the Sides; and DE, CE, are the Segments of the Base whose Difference is DF; because (by Cor. Prop. 38. El. 3.) the Rectangle under DC and DF, is equal to the Rectangle under DG, DH, it shall be (by 16 El. 6.), as $DC : DG :: DH : DF$.

PROBLEM.

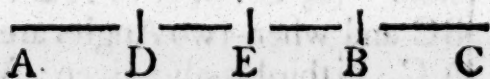
The Sum and Difference of any two Quantities being given, to find the Quantities themselves.

IF one Half of the Sum be added to one Half of the Difference, the Aggregate shall be equal to the greater of the Quantities; and if from one Half of the Sum be taken one Half of the Difference, the Residue shall be equal to the lesser of the Quantities. For, let there be two Quantities AB, BC, and let there be taken $AD = BC$; then DB will be their Difference, and AC their Sum; which, bisected in E, gives AE or EC the half Sum; and DE or EB the half Difference. Hence, $AB = AE + EB =$ the half Sum + the half Difference, and $BC = CE - EB =$ the half Sum - the half Difference.

In any plain Triangle, if two Angles be given, the third Angle is also given; because it is their Complement to two right Angles.

If one of the acute Angles of a right-angled Triangle be given, the other acute Angle will be given, because it is the Complement of the given Angle to a right Angle.

And if two Sides of a right-angled Triangle be given, the other Side may be found by the first Proposition without a Canon.



The

The Trigonometrical Solutions of a right-angled Triangle, may be as follow. Vid. Fig. A.

	Given	Sought	Make as
1	The Legs A B and B C.	The Angles.	$AB : BC :: R : T$ of the Angle A, whose Complement is the Angle C.
2	The Leg A E, and the Hypo- thenuse A C.	The Angles.	$AC : AB :: R : S, C,$ whose Complement is the Angle A.
3	The Leg A B, and the Angle A.	The other Side B C, and the Hypo- thenuse A C.	$R : T, A :: AB : BC.$ $S, C : R :: AB : AC.$
4	The Hypo- thenuse A C, and the Angle C.	The Leg A B.	$R : S, C :: AC : AB.$

The Trigonometrical Solutions of oblique-angled Triangles. Vid. Fig. to Prop. 12.

	Given	Sought	Make as
1	The Angles A, B, C, and the Side AB.	The Sides BC and A C.	$S, C : SA :: AB : BC.$ Also, $S, C : S, B :: AB : AC.$ But when two Angles are given, the third is also given: Whence the Case wherein two Angles and a Side are given, to find the rest, falls into this Case.

Given

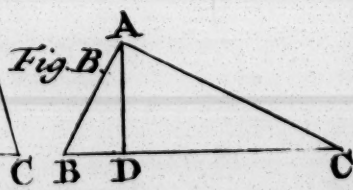
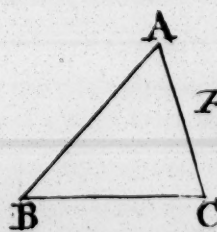
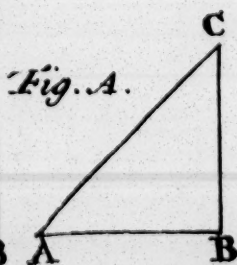
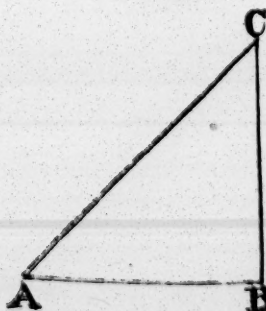
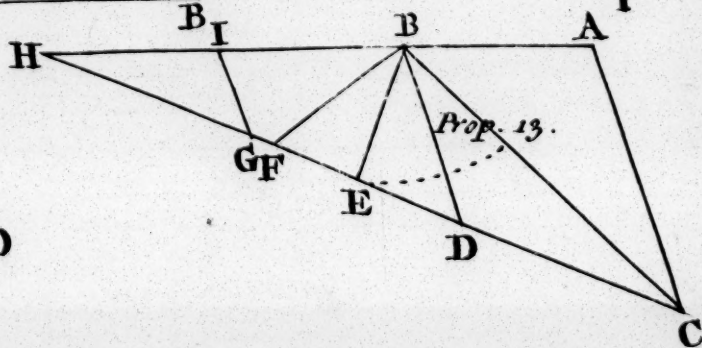
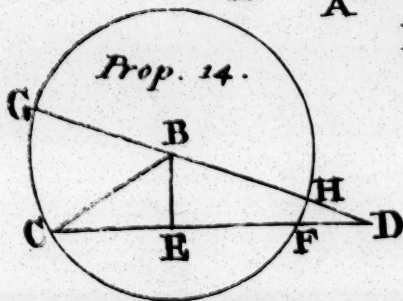
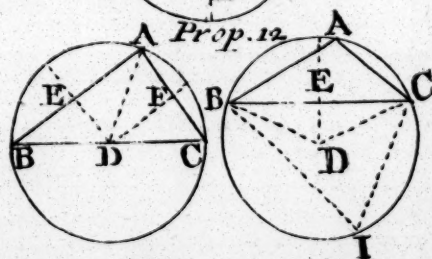
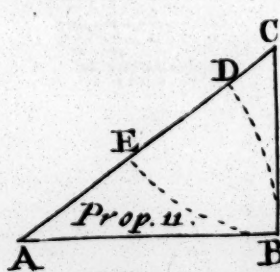
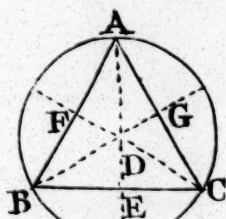
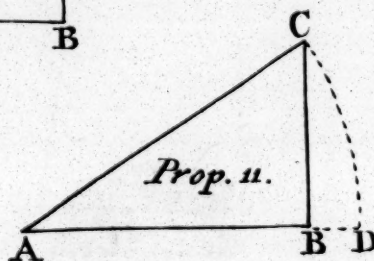
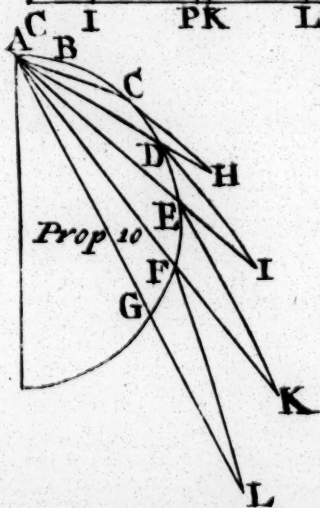
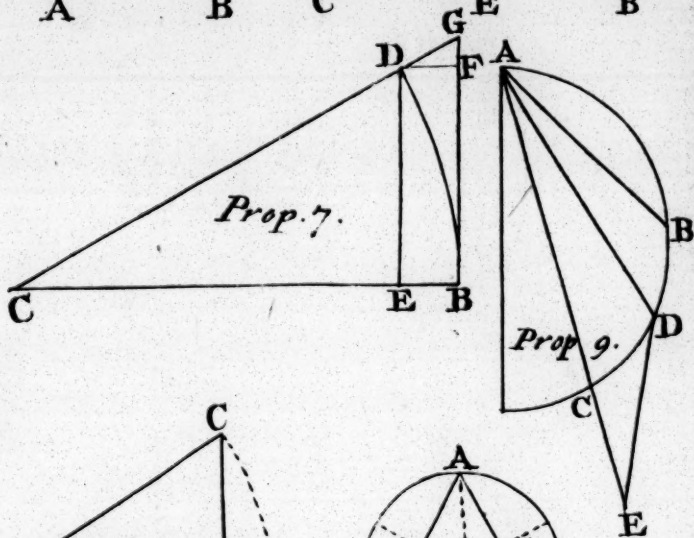
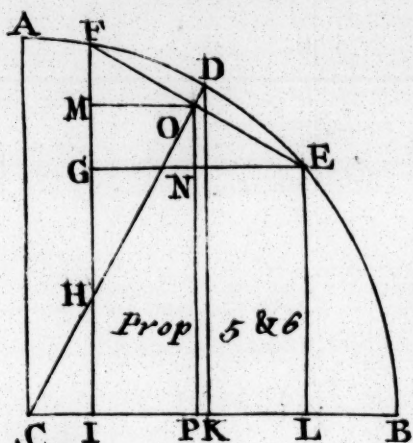
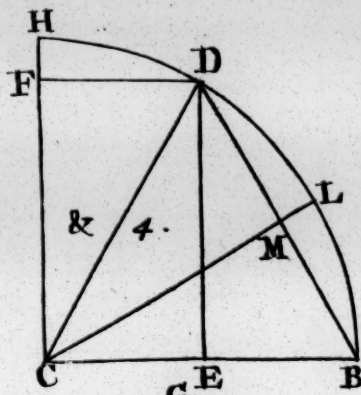
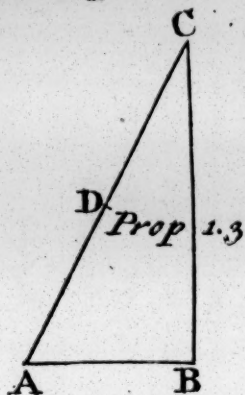
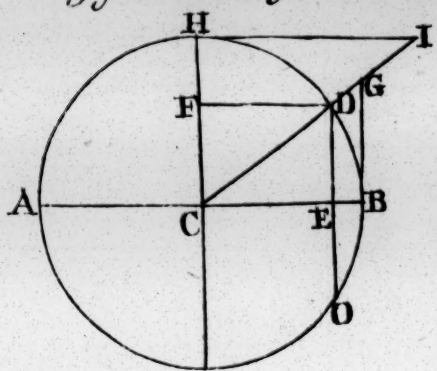
Given	Sought	Make as
2 All the Angles A, B, C.	All the Sides AB, AC, BC.	$S, C : S, A :: AB : BC$. And $S, C : S, B :: AB : AC$. Whence, if the Angles are given, the Proportions of the Sides may be found, but not the Sides themselves, unless one of them be first known.
3 The two Sides AB, BC, and C, the Angle opposite to one of them.	The Angles A and B.	$AB : BC :: S, C : S, A$; which therefore may be found. When A B, the Side opposite to C, the given Angle, is longer than CB, the Side opposite to the sought Angle, the sought Angle is less than a right one. But when it is shorter, because the Sine of an Angle, and that of its Complement to two right Angles, is the same, the Species of the Angle A must be first known, or the Solution will be ambiguous.
4 The two Sides AB, BC, and the interja-cent Angle B.	The Angles A and C.	<i>Vid. Fig. to Prop. 13.</i> $BC + AB : BC - AB :: T, \frac{A + C}{2} T, \frac{A - C}{2}$ Whence is known the Difference of the Angles A and C, whose Sum is given; and so (by the <i>Prob.</i> following <i>Prop. 14.</i>) the Angles themselves will be given.
5 All the Sides AB, BC, AC.	The Angles.	<i>Vid. Fig. B.</i> Let the Perpendicular be drawn from the Vertex to the Base, and find the Segments of the Base by <i>Prop. 14.</i> viz. make as $BC : AC + AB :: AC - AB : DC - DB$. And so BD, DC, are given from this Analogy; and thence the Angles ABD, ADC, will be given by the Resolution of right-angled Triangles.

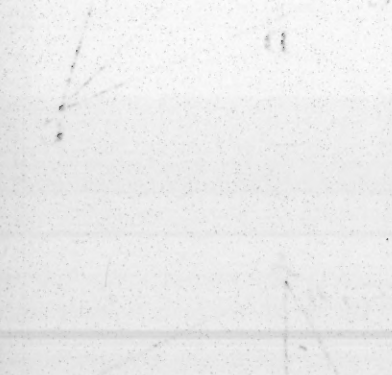
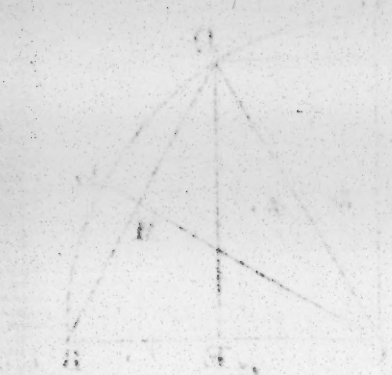
T H E
E L E M E N T S
O F
Spherical Trigonometry.

D E F I N I T I O N S.

- I. **T**HE Poles of a Sphere are two Points in the Superficies of the Sphere, that are the Extremes of the Axis.
- II. The Pole of a Circle in a Sphere is a Point in the Superficies of the Sphere, from which all Right Lines that are drawn to the Circumference of the Circle are equal to one another.
- III. A great Circle in a Sphere, is that whose Plane passes through the Centre of the Sphere, and whose Centre is the same of that of the Sphere.
- IV. A spherical Triangle is a Figure comprehended under the Arcs of three great Circles in a Sphere.
- V. A spherical Angle is that which, in the Superficies of the Sphere, is contained under two Arcs of great Circles; and this Angle is equal to the Inclination of the Planes of the said Circles.

Fig. for the Definitions and Prop. 2.





PROPOSITION I.

Great Circles ACB, AFB, mutually bisect each other.

FOR since the Circles have the same Centre, their common Section shall be a Diameter of each Circle, and so will cut them into two equal Parts.

Coroll. Hence the Arcs of two great Circles in the Superficies of the Sphere, being less than Semicircles, do not comprehend a Space; for they cannot, unless they meet each other in two opposite Points in a Semicircle.

PROPOSITION II.

If from the Pole C of any Circle AFB, be drawn a right Line CD to the Centre thereof, the said Line will be perpendicular to the Plane of that Circle. Vid. Fig. to Prop. I.

LET there be drawn any Diameter EF, GH, in the Circle AFB; then, because the Triangles CDF, CDE, the Sides CD, DF, are equal to the Sides CD, DE, and the Base CF equal to the Base CE (*by Def. 2.*); then (*by 4 El. 1.*) shall the Angle CDF = Angle CDE, and so each of them will be a right Angle. After the same manner we demonstrate, that the Angles CDG, CDH, are right Angles; and so (*by 4 El. 11.*) CD shall be perpendicular to the Plane of the Circle AFE. W. W. D.

Coroll. 1. A great Circle is distant from its Pole by the Interval of a Quadrant; for since the Angles CDG, CDF, are right Angles, the Measures of them, *viz.* the Arcs CG, CF, will be Quadrants.

2. Great Circles, that pass thro' the Pole of some other Circle, make right Angles with it; and contrariwise, if great Circles make right Angles with some other Circle, they shall pass thro' the Poles of that other Circle; for they must necessarily pass thro' the right Line DC.

PROPOSITION III.

If a great Circle ECF be described about the Pole A; then the Arc CF intercepted between AC, AF, is the Measure of the Angle CAD or CDF. Vid. Fig. to Prop. 1.

THE Arcs AC, AF, (by Cor. 1. Prop. 2.) are Quadrants; and, consequently, the Angles ADC, ADF, are right Angles: Wherefore, (by Def. 6. El. 11.) the Angle CDF (whose Measure is the Arc CF) is equal to the Inclination of the Planes ACB, AFB, and also equal to the spherical Angle CAF or CBF. W.W.D.

Coroll. 1. If the Arcs AC, AF, are Quadrants, then shall A be the Pole of the Circle passing thro' the Points C and F; for AD is at right Angles to the Plane FDC (by 14 El. 11.).

2. The vertical Angles are equal, for each of them is equal to the Inclination of the Circles; also the adjoining Angles are equal to two right Angles.

PROPOSITION IV.

Triangles shall be equal and congruous, if they have two Sides equal to two Sides, and the Angles comprehending the two Sides also equal.

PROPOSITION V.

Also Triangles shall be equal and congruous, if one Side, together with the adjacent Angles in one Triangle, be equal to one Side, and the adjacent Angles of the other Triangle.

PROPOSITION VI.

Triangles mutually Equilateral, are also mutually Equiangular.

PROPOSITION VII.

In Isosceles Triangles, the Angles at the Base are equal.

P R O

PROPOSITION VIII.

And if the Angles at the Base be equal, then the Triangle shall be Iſoſceles.

THESE four laſt Propoſitions are demonſtrated in the ſame manner as in plain Triangles.

PROPOSITION IX.

Any two Sides of a Triangle are greater than the third.

FOR the Arc of a great Circle is the ſhorteſt Way, between any two Points in the Superficies of the Sphere.

PROPOSITION X.

A Side of a ſpherical Triangle is leſs than a Semicircle.

LET AC, AB, the Sides of the Triangle ABC, be produced till they meet in D; then ſhall the Arc ACD, which is greater than the Arc AC, be a Semicircle.

PROPOSITION XI.

The three Sides of a ſpherical Triangle are leſs than a whole Circle.

FOR $BD + DC$ is greater than BC (by Prop. 9.); and adding on each Side $BA + AC$, $DBA + DCA$; that is, a whole Circle will be greater than $AB + BC + AC$, which are the three Sides of the ſpherical Triangle ABC.

PROPOSITION XII.

In any ſpherical Triangle ABC, the greater Angle A is ſubtended by the greater Side.

MAKE the Angle $BAD = \text{Angle } B$; then ſhall $AD = BD$ (by 8 of this); and ſo $BDC = DA + DC$, and theſe Arcs are greater than AC . Wherefore the Side BC , that ſubtends the Angle BAC , is greater than the Side AC , that ſubtends the Angle B .

PROPOSITION XIII.

In any spherical Triangle ABC, if the Sum of the Legs AB and BC be greater, equal, or less, than a Semicircle, then the internal Angle at the Base AC shall be greater, equal, or less, than the external and opposite Angle BCD; and so the Sum of the Angles A and ACB shall also be greater, equal, or less, than two right Angles.

FIRST, let $AB + BC = \text{Semicircle} = AD$, then shall $BC = BD$, and the Angles BCD and D equal (*by 8 of this*); and therefore the Angle BCD shall be $=$ Angle A.

Secondly, Let $AB + BC$ be greater than ABD ; then shall BC be greater than BD ; and so the Angle D (that is, the Angle A, *by 12 of this*) shall be greater than the Angle BCD. In like manner we demonstrate, if $AB + BC$ be together less than a Semicircle, that the Angle A will be less than the Angle BCD: And because the Angles BCD and BCA are $=$ two right Angles, if the Angle A be greater than the Angle BCD, then shall A and BCA be greater than two right Angles; if the Angle A $=$ BCD, then shall A and BCA be equal to two right Angles; and if A be less than BCD, then will A and BCA be less than two right Angles. W. W. D.

PROPOSITION XIV.

In any spherical Triangle GHD, the Poles of the Sides, being joined by great Circles, do constitute another Triangle XMN, which is the Supplement of the Triangle GHD; viz. the Sides NX, XM, and NM, shall be Supplements of the Arcs that are the Measures of the Angles D, G, H, to the Semicircles; and the Arcs that are the Measures of the Angles M, X, N, will be the Supplements of the Sides GH, GD, and HD, to Semicircles.

FROM the Poles G, H, D, let the great Circles XCAM, TMNO, XKBN, be describ'd; then, because

because G is the Pole of the Circle $XCAM$, we shall have $GM = \text{Quadrant}$ (*by Cor. 1. Prop. 2.*); and since H is the Pole of the Circle TMO , then will HM be also a Quadrant; and so (*by Cor. 1. Prop. 3.*) M shall be the Pole of the Circle GH . In like manner, because D is the Pole of the Circle XBN , and H the Pole of the Circle TMN , the Arcs DN , HN , will be Quadrants; and so (*by Cor. 1. Prop. 3.*) N shall be the Pole of the Circle HD . And because GX , DX , are Quadrants, X will be the Pole of the Circle GD . These Things premis'd,

Because $NK = \text{Quadrant}$, (*by Cor. 1. Prop. 2.*) then will $NK + XB$, that is, $NX + KB = \text{two Quadrants}$, or a Semicircle; and so NX is the Supplement of the Arc KB , or of the Measure of the Angle HDG to a Semicircle. In like manner, because $MC = \text{Quadrant}$, and $XA = \text{Quadrant}$, then will $MC + XA$; that is, $XM + AC = \text{two Quadrants}$ or Semicircle; and, consequently, XM is the Supplement of the Arc AC , which is the Measure of the Angle HGD . Likewise, since MO , NT , are Quadrants, we shall have $MO + NT = OT + NM = \text{Semicircle}$: And therefore NM is the Supplement of the Arc OT , or of the Measure of the Angle GHD to a Semicircle. W. W. D.

Moreover, because DK , HT , are Quadrants, $DK + HT$, or $KT + HD$, are equal to two Quadrants, or a Semicircle; therefore KT , or the Measure of the Angle XNM , is the Supplement of the Side HD to a Semicircle. After the same manner it is demonstrated, that OC , the Measure of the Angle XMN , is the Supplement of the Side GH ; and BA the Measure of the Angle X , is the Supplement of the Side GD . W. W. D.

PROPOSITION XV.

Equiangular spherical Triangles are also equilateral.

FOR their Supplementals (*by 14 of this*) are equilateral, and therefore equiangular also; and so themselves are likewise equilateral (*by Part 2. Prop. 14.*).

PROPOSITION XVI.

The three Angles of a spherical Triangle are greater than two right Angles, and less than six.

FOR the three Measures of the Angles G, H, D, together with the three Sides of the Triangle XNM, make three Semicircles (*by 14 of this*); but the three Sides of the Triangle XNM are less than two Semicircles (*by 11 of this*). Wherefore the three Measures of the Angles G, H, D, are greater than a Semicircle; and so the Angles G, H, D, are greater than two right Angles.

The second Part of the Proposition is manifest; for in every spherical Triangle, the external and internal Angles together, only make six right Angles: Wherefore the internal Angles are less than six right Angles.

PROPOSITION XVII.

If from the Point R, not being the Pole of the Circle AFBE, there fall the Arcs RA, RB, RG, RV, of great Circles to the Circumference of that Circle; then the greatest of those Arcs is RA, which passes through the Pole C thereof; and the Remainder of it is the least; and those that are more distant from the greatest are less than those which are nearer to it, and they make an obtuse Angle with the former Circle AFB, on the Side next to the greatest Arc. Vid. Fig. to Prop. 1.

BEcause C is the Pole of the Circle AFB, then shall CD, and RS, which is parallel thereto, be perpendicular to the Plane AFB. And if SA, SG, SV, be drawn, then shall SA (*by 7 El. 3.*) be greater than SG, and SG greater than SV. Whence, in the right-angled plain Triangles RSA, RSG, RSV, we shall have $RSq + SAq$, or RAq , greater than $RSq + SGq$, or RGq ; and so RA will be greater than RG, and the Arc RA greater than the Arc RG. In like manner, $RSq + SGq$, or RGq , shall be greater than

than $\widehat{RSq} + \widehat{SVq}$, or $\widehat{RVq}$; and so $\widehat{RG}$ shall be greater than $\widehat{RV}$, and the Arc $\widehat{RG}$ greater than the Arch $\widehat{RV}$.

2dly, The Angle $\widehat{RGA}$ is greater than the Angle $\widehat{CGA}$, which is a right Angle (*by Cor. Prop. 3.*); and the Angle $\widehat{RVA}$ is greater than the Angle $\widehat{CVA}$, which also is a right Angle. Therefore the Angles $\widehat{RGA}$, $\widehat{RVA}$, are obtuse Angles.

PROPOSITION XVIII.

In a spherical Triangle, right-angled at A, the Legs containing the right Angle are of the same Affection with the opposite Angles; that is, if the Legs be greater or less than Quadrants, then, accordingly, will the Angles opposite to them be greater or less than right Angles. Vid. Fig. to Prop. 1.

FOR if $\widehat{AC}$ be a Quadrant, then will C be the Pole of the Circle $\widehat{AFB}$; and the Angles $\widehat{AGC}$, or $\widehat{AVC}$, will be right Angles. If the Leg $\widehat{AR}$ be greater than a Quadrant, then shall the Angle $\widehat{AGR}$ be greater than a right Angle (*by 17 of this*); and if the Leg $\widehat{AX}$ be less than a Quadrant, the Angle $\widehat{AGX}$ shall be less than a right Angle.

PROPOSITION XIX.

If two Legs of a right-angled spherical Triangle be of the same Affection, (and consequently the Angles) that is, if they are both less or both greater than a Quadrant, then will the Hypotenuse be less than a Quadrant. Vid. Fig. to Prop. 1.

IN the Triangle $\widehat{ARV}$, or $\widehat{BRV}$, let F be the Pole of the Leg $\widehat{AR}$; then will $\widehat{RF}$ be a Quadrant, which is greater than $\widehat{RV}$ (*by 17 of this*).

PROPOSITION XX.

If they be of a different Affection, then shall the Hypotenuse be greater than a Quadrant. Vid. Fig. to Prop. 1.

FOR in the Triangle $\widehat{ARG}$, the Hypotenuse $\widehat{RG}$ is greater than $\widehat{RF}$, which is a Quadrant.

PRO-

PROPOSITION XXI.

If the Hypothenuſe be greater than a Quadrant, then the Legs of the right Angle, and ſo the Angles oppoſite to them, are of a different Affection; but if leſſer, of the ſame Affection. Vid. Fig. to Prop. I.

THIS Propoſition, being the Converſe of the former ones, eaſily follows from them.

PROPOSITION XXII.

In any ſpherical Triangle ABC, if the Angles at the Baſe B and C, be of the ſame Affection, then the Perpendicular falls within the Triangle; and if they be of a different Affection, the Perpendicular falls without the Triangle.

IN the firſt Caſe, if the Perpendicular does not fall within, let it fall without the Triangle, (as in *Fig. 2.*) then in the Triangle ABP, the Side AP is of the ſame Affection with the Angle B. And, in like manner, in the Triangle ACP, AP is of the ſame Affection with the Angle ACP; therefore, ſince ABC, and ACP, are of the ſame Affection, the Angles ABC, ACB, ſhall be of a different Affection; which is contrary to the Hypotheſis.

In the ſecond Caſe, if the Perpendicular does not fall without, let it fall within (as in *Fig. 1.*). Then, in the Triangle ABP, the Angle B is of the ſame Affection with the Leg AP. So, likewise, in the Triangle ACP, the Angle C is of the ſame Affection with AP; and therefore the Angles B and C are of the ſame Affection; which is contrary to the Hypotheſis.

PROPOSITION XXIII.

In ſpherical Triangles BAC, BHE, right-angled at A and H, if the ſame acute Angle B be at the Baſe BA, or BH, then the Sines of the Hypothenuſes ſhall be proportional to the Sines of the perpendicular Arcs.

FOR the right Lines CD, EF, being perpendicular to the ſame Plane, are parallel. Alſo, FR, DP,

DP, perpendicular to the Radius OB, are likewise parallel: Wherefore the Planes of the Triangles EFR, CDP, are also parallel (*by 15 El. 11.*). Wherefore CP, ER, the common Sections of those Planes, with the Plane passing thro' BE, CO, will be parallel (*by 16 El. 11.*). Therefore the Triangles CDP, EFR, shall be equiangular: Wherefore CP, the Sine of the Hypothenufe BC, is to CD the Sine of the perpendicular Arc CA, as ER the Sine of the Hypothenufe BE, to EF the Sine of the perpendicular Arc EH. W. W. D.

PROPOSITION XXIV.

The same Things being supposed, AQ, HK, the Sines of the Bases, are proportional to IA, GH, the Tangents of the perpendicular Arcs.

FOR, after the same manner as in the last Proposition, we demonstrate, that the Triangles QAI, KHG, are equiangular; whence, $QA : AI :: KH : HG$.

PROPOSITION XXV.

In a spherical Triangle ABC, right-angled at A, as the Cosine of the Angle B, at the Base BA, is to the Sine of the vertical Angle ACB, so is the Cosine of the Perpendicular to the Radius.

PREPARATION.

LET the Sides AB, BC, CA, be produced, so that BE, BF, CI, CH, be Quadrants; and from the Poles B and C draw the great Circles EFDG, IHG; then will the Angles at E, F, I, H, be right Angles; and so D is the Pole of BAE, *by Cor. 2. Prop. 2. of this*) and G the Pole of IFCB: Also, AE will be = Complement of the Arc BA, and FE the Measure of the Angle B = GD, and DF their Complement: Also, BC shall be = FI = Measure of the Angle G, and CF their Complement: Likewise, CA = HD, and DC their Complement. These Things premis'd, in the Triangles HIC, DCF, right-angled at I and F, and having the same acute Angle C; since BA is less than a Quadrant, it will be, as S. DF:

DF : S, HI :: S, DC : S, HC; that is, the Cosine of the Angle B is to the Sine of the vertical Angle BCA, as the Cosine of CA is to Radius. W. W. D.

PROPOSITION XXVI.

*The Cosine of the Base : Cosine of the Hypothenuse ::
R : Cos. of the Perpendicular.*

FOR in the Triangles AED, CFD, right-angled at E, F, having the same acute Angle D; because AE is less than a Quadrant, we have S, EA : S, CF :: S, DA : S, DC. W. W. D.

PROPOSITION XXVII.

*S, of the Base : R :: T, of the Perpendicular : T,
of the Angle at the Base.*

FOR in the Triangles BAC, BEF, right-angled at A and E, and having the same acute Angle B; because AC is less than a Quadrant, we have S, BA : S, BE :: T, AC : T, EF. W. W. D.

PROPOSITION XXVIII.

Cos. of the vertical Angle : R :: T, of the Perpendicular : T, of the Hypothenuse.

IN the Triangles GIF, GHD, right-angled at I and H, and having the same acute Angle G, because HD is less than HC, or a Quadrant, it is, as S, GH : S, GI :: T, HD : T, IF.

PROPOSITION XXIX.

S, of the Hypothenuse : R :: S, of the Perpendicular : S, of the Angle at the Base.

IN the afore said Triangles, we have S, IF : S, GF :: S, HD : S, GD.

PROPOSITION XXX.

$R : \text{Cof. of the Hypotenuse} :: T, \text{ of the vertical Angle} : \text{Cot. of the Angle at the Base.}$

IN the Triangles HIC, DFC , right-angled at I and F , and having the same acute Angle C , because DF is less than a Quadrant, we have $S, CI : S, CF :: T, HI : T, DF$; that is, $R : \text{Cof. } BC :: T, C : \text{Cot. } B$.

The last six Propositions are sufficient for solving all the sixteen Cases of right-angled spherical Triangles. These sixteen Cases, with their Analogies deduced from the said Propositions, are as follow.

	Given besides the Right Angle	Sought	<i>Id. Fig. to Prop. 25, 26, 27, 28, 29, 30.</i>	
I	AC and C .	B .	$R : \text{Cof. } CA :: S, C : \text{Cof. } B$ of the same Kind with CA .	By the Inverse of Prop. 25.
2	AC and B .	C .	$\text{Cof. } CA : R :: \text{Cof. } B : S, C$; this is ambiguous.	by Prop. 25.
3	B and C .	AC .	$S, C : \text{Cof. } B :: R : \text{Cof. } CA$, of the same Kind with the Angle B .	by Prop. 25, and 18.
4	BA, CA .	BC .	$R : \text{Cof. } BA :: \text{Cof. } AC : \text{Cof. } BC$. If BA, AC , be of the same Affection, and not Quadrants, then BC will be less than a Quadrant. If they be of a different Affection, BC shall be greater than a Quadrant.	by Prop. 26, 19, and 20.

Given

	Given besides the Right Angle	Sought		
5	BA, BC.	AC.	Cof. BA : R :: Cof. BC : Cof. CA. If BC be less than a Quadrant, then shall BA and CA be of the same Affection; if greater, of a different: But BA is given, and therefore the Species thereof. Wherefore the Species of AC is also given.	By Prop. 26, and 21.
6	BA, CA.	B.	S, BA : R :: T, CA : T, B of the same Affection with the opposite Side CA.	by Prop. 27, and 18.
7	BA, B.	AC.	R : S, BA :: T, B : T, AC of the same Kind with B.	by Prop. 27, and 18.
8	AC, B.	BA.	T, B : R :: T, CA : S, BA ambiguous.	by Prop. 27.
9	BC, C.	AC.	R : Cof. C :: T, BC : T, CA. If BC be less than a Quadrant, and the Angles C and B are of the same Affection; if greater, of a different. Therefore, if the Species of the Angle B be given, then will AC be given.	by Prop. 28, and 21.
10	AC, C.	BC.	Cof. C : R :: T, CA : T, BC. And so, if the Angle C, and CA, be of the same Affection, then BC shall be lesser than a Quadrant; if of a different, greater.	by Prop. 28, 20, and 21.
11	BC, AC.	C.	T, BC : R :: T, CA : Cof. C. If BC be less than a Quadrant, then CA and BA, and consequently the Angles, shall be of the same Affection; if greater, of a different, but the Species of CA is given. Therefore the Species of the Angle C will be also given.	by Prop. 28, and 21.

	Given beside the Right Angle.	Sought		
12	BC, B.	AC.	$R : S, BC :: S, B : S, AC$ of the same Species with B.	By Prop. 29, and 18.
13	AC, B.	BC.	$S, B : S, AC :: R : S, BC$ ambi- guous.	by Prop. 29.
14	BC, AC.	B.	$S, BC : R :: S, AC : S, B$ of the same Species with CA.	by Prop. 29.
15	B, C.	BC.	$T, C : R :: \text{Cot. } B : \text{Cof. } BC$. And so, if the Angles B and C are of the same Affection, then shall BC be less than a Quadrant; if of a different, greater.	by Prop. 30, 19, and 20.
16	BC, C.	B.	$R : \text{Cof. } BC :: T, C : \text{Cot. } B$. And so if BC be lesser than a Quadrant, the Angles C and B shall be of the same Affection; if greater, of a different. But the Species of the Angle C is given; therefore the Species of the Angle B will be given also.	by Prop. 30, and 31.

Of

Of the Solution of right-angled spherical Triangles, by the five circular Parts.

THE Lord Napier, (the noble Inventor of Logarithms) by a due Consideration of the Analogies by which right-angled spherical Triangles are solv'd, found out two Rules, easy to be remember'd, by means of which, all the sixteen Cases may be solv'd: For since, in these Triangles, besides the right Angle, there are three Sides, and two Angles; the two Sides comprehending the right Angle, and the Complements of the Hypothenufe, and the two other Angles, were call'd, by Napier, *Circular Parts*; and then there are given any two of the said Parts, and a third is sought. One of these three, which is call'd the *Middle Part*, either lies between the other two Parts, which are call'd *Adjacent Extremes*, or is separated from them, and then are call'd *Opposite Extremes*: So if the Complement of the Angle B (*Fig. to Prop. 25.*) be suppos'd the middle Part, then the Leg AB, and the Complement of the Hypothenufe BC, are adjacent extreme Parts; but the Complement of the Angle C, and the Side AC, are opposite Extremes. Also, if the Complement of the Hypothenufe BC be suppos'd the middle Part, then the Complements of the Angles B and C are adjacent Extremes, and the Legs AB, AC, are opposite Extremes. In like manner, supposing the Leg AB the middle Part, the Complement of the Angle B and AC are adjacent Extremes; for the right Angle A does not interrupt the Adjacency, because it is not a circular Part. But the Complement of the Angle C, and the Complement of the Hypothenufe BC, are opposite Extremes to the said middle Part. These Things premis'd,

R U L E I.

In any right-angled spherical Triangle, the Rectangle under the Radius, and the Sine of the middle Part, is equal to the Rectangle under the Tangents of the adjacent Parts.

R U L E

RULE II.

The Rectangle under the Radius, and the Sine of the middle Part, is equal to the Rectangle under the Cosines of the opposite Parts.

Each of the Rules have three Cases : For the middle Part may be the Complement of the Angle B, or C, or the Complement of the Hypothenuse BC ; or one of the Legs, AB, AC.

Case 1. Let the Complement of the Angle C be the middle Part ; then shall AC, and the Complement of the Hypothenuse BC, be adjacent Extremes. By *Prop. 28.* the Cosine of the vertical Angle C is to Radius, as the Tangent of CA is to the Tangent of the Hypothenuse BC. Then (by Alternation) we shall have $\text{Cof. } C : T, CA :: R : T, BC$. But $R : T, BC :: \text{Cot. } BC : R$ (as has been before shewn). Wherefore $\text{Cof. } C : T, AC :: \text{Cot. } BC : R$; whence $R \times \text{Cof. } C = T, AC \times \text{Cot. } BC$.

And the Complement of the Angle B, and AB, are opposite Extremes, to the same middle Part, the Complement of the Angle C ; and, (by *Prop. 25.*) as the Cosine of the Angle C, to the Sine of the Angle CDF, so is the Cosine of DF to Radius. But the Sine of $CDF = AE = \text{Cof. } BA$, and $\text{Cof. } DF = S, EF = S, \text{Angle } B$. Whence it will be, as $\text{Cof. } C : \text{Cof. } BA :: S, B : R$. And $R \times \text{Cof. } C = \text{Cof. } BA \times S, B$; that is, Radius drawn into the Sine of the middle Part, is equal to the Rectangle under the Cosines of the opposite Extremes.

Case 2. Let the Complement of the Hypothenuse BC be the middle Part ; then the Complements of the Angles B and C will be adjacent Extremes. In the Triangle DCF (by *Prop. 27.*) it is as $S, CF : R :: T, DF : T, C$. Whence (by Alternation) $S, CF : T, DF :: (R : T, C ::) \text{Cot. } C : R$. But $S, CF = \text{Cof. } BC$, and $T, DF = \text{Cot. } B$: Wherefore $R \times \text{Cof. } BC = \text{Cot. } C \times \text{Cot. } B$; that is, Radius drawn into the Sine of the middle Part, is equal to the Product of the Tangents of the adjacent extreme Parts.

X

And

And BA, AC, are the opposite Extremes to the said middle Part, *viz.* the Complement of BC; and (*by Prop. 26.*) $\text{Cof. BA} : \text{Cof. BC} :: R : \text{Cof. AC}$. Wherefore we shall have $R \times \text{Cof. BC} = \text{Cof. BA} \times \text{Cof. AC}$.

Case 3. Lastly, Let AB be the middle Part; and then the Complement of the Angle B, and AC, will be adjacent Extremes, and (*by Prop. 27.*) $S, AB : R :: T, CA : TB$. Whence, $S, AB : T, CA :: (R : T, B ::) \text{Cot. B} : R$. And so $R \times S, AB = T, CA \times \text{Cot. B}$.

Moreover, the Complement of BC, and the Angle C, are opposite Extremes to the same middle Part AB; and in the Triangles GHD (*by Prop. 25.*) we have $\text{Cof. D} : S, DGH :: \text{Cof. GH} : R$. But $\text{Cof. D} = \text{Cof. AE} = S, AB$, and $S, G = S, IF = S, BC$. Also, $\text{Cof. GH} = S, HI = S, C$. Wherefore it will be, as $S, AB : S, BC :: S, C : R$. And hence $R \times S, AB = S, BC \times S, C$.

And so, in every Case, the Rectangle under the Radius, and the Sine of the middle Part, shall be equal to the Rectangle under the Cosines of the opposite Extremes, and to the Rectangle under the Tangents of the adjacent Extremes: And, consequently, if the aforesaid Equations be resolv'd into Analogies, (*by 16 El. 6.*) the unknown Parts may be found by the Rule of Proportion. And if the Part sought be the middle one, then shall the first Term of the Analogy be Radius, and the second and third the Tangents or Cosines of the extreme Parts. If one of the Extremes be sought, the Analogy must begin with the other; and the Radius, and the Sine of the middle Part, must be put in the middle Places, that so the Part sought may be in the fourth Place.

IN oblique-angled spherical Triangles (*Fig. to Prop. 31.*) BCD, if a perpendicular Arc AC be let fall from the Angle C to the Base, continu'd, if need be, so as to make two right-angled spherical Triangles BAC, DAC; then, by those right-angled Triangles, may most of the Cases of oblique-angled ones be solv'd.

PROPOSITION XXXI.

The Cosines of the Angles B and D, at the Base BD, are proportional to the Sines of the vertical Angles BCA, DCA.

FOR $\text{Cof. Angle } B : S, BCA :: (\text{Cof. } CA : R ::)$
 $\text{Cof. } D : S, DCA$ (by 25 of this).

PROPOSITION XXXII.

The Cosines of the Sides BC, DC, are proportional to the Cosines of the Bases BA, DA.

FOR $\text{Cof. } BC : \text{Cof. } BA :: (\text{Cof. } CA : R ::)$
 $\text{Cof. } DC : \text{Cof. } DA$ (by 26 of this).

PROPOSITION XXXIII.

The Sines of the Bases BA, DA, are in a reciprocal Proportion of the Tangents of the Angles B and D at the Base BD.

BECAUSE (by 27 of this) $S, BA : R :: T, AC : T,$
of the Angle B. And by the same inversely, $R : S, DA :: T,$
of the Angle D : T, AC . Then will it be
(by the Equality of perturbate Ratio, according to
Prop. 23. El. 5.) $S, BA : S, DA :: T,$ Angle D : $T,$
Angle B.

PROPOSITION XXXIV.

The Tangents of the Sides BC, DC, are in a reciprocal Proportion of the Cosines of the vertical Angles BCA, DCA.

BECAUSE by alternating the 28th Proposition, we have

$T, BC : R :: T, CA : \text{Cof. } BCA,$
and by the same $R : \text{Cof. } DCA :: T, DC : T, CA.$
Wherefore, by Equality of perturbate Proportion,
 $T, BC : \text{Cof. } DCA :: T, DC : \text{Cof. } BCA.$

PROPOSITION XXXV.

The Sines of the Sides BC, DC, are proportional to the Sines of the opposite Angles D and B.

BEcause, by the 29th of *this*, $S, BC : R :: S, CA : S, \text{ of the Angle } B$; and, by the same, inverting, $R : S, DC :: S, \text{ Angle } D : S, \text{ of } CA$: Whence, by Equality of perturbate Ratio, $S, BC : S, DC :: S, D : S, B$.

PROPOSITION XXXVI.

In any spherical Triangle ABC, the Rectangle $CF \times AE$, or $FM \times AE$, contained under the Sines of the Legs BC, BA, is to the Square of the Radius, as IL or $IA - LA$ the Difference of the versed Sines of the Base CA, and the Difference of the Legs AM, to GN , the versed Sine of the Angle B.

LET a great Circle PN be describ'd from the Pole B ; and let BP, BN , be Quadrants; and then PN is the Measure of the Angle B ; also, describe from the same Pole B a lesser Circle CFM thro' C ; the Planes of these Circles shall be perpendicular to the Plane $BO N$ (by the 2d of *this*). And PG, CH , being perpendicular in the same Plane, fall on the common Sections ON, FM ; suppose in G, H . Again, draw HI , perpendicular to AO ; and then the Plane drawn thro' CH, HI , shall be perpendicular to the Plane AOB . Whence AI , which is perpendicular to HI , will be perpendicular to the right Line CI (by *Def. 4. El. 11.*); and so AI is the versed Sine of the Arc AC , and AL the versed Sine of the Arc $AM = BM - BA = BC - BA$. The Isosceles Triangles CFM, PON , are equiangular, since MF, NO , as also CF, PO , (by 16 *El. 11.*) are parallel. Wherefore, if Perpendiculars CH, PG , be drawn to the Sides FM, ON , the Triangles will be divided similarly, and we shall have $FM : ON :: MH : GN$. Also, because the Triangles AOE, DIH, DLM , are equiangular, we shall have $AE : AO :: IL : MH$.

But

But it has been prov'd, that $FM : ON :: MH : GN$.
Wherefore it shall be, as $AE \times FM : AO \times ON ::$
 $IL \times MH \times MH \times GN$, or as IL to GN ; that is,
the Rectangle under the Sines of the Legs, is to the
Square of Radius, as the Difference of the versed Sines
of the Base, and the Difference of the Legs BC, BA ,
is to the versed Sine of the Angle B . W. W. D.

PROPOSITION XXXVII.

*The Difference of the versed Sines of two Arcs
drawn into half the Radius, is equal to the
Rectangle under the Sine of half the Sum and
the Sine of half the Difference of those Arcs.*

LET there be two Arcs, BE, BF , whose Differ-
ence EF let be bisected in D ; then shall BD
be the half Sum, and FD the half Difference of those
Arcs. $GE = IL$ is the Difference of the versed Sines
of the Arcs BE, BF ; also, FO is the Sine of the half
Differences of the Arcs. And because the Triangles
 CDK, FEG , are equiangular, we have $DK : GE ::$
 $(CD : FE ::) \frac{1}{2} CD : \frac{1}{2} FE$. Whence $DK \times \frac{1}{2} FE$, or
 $DK \times FO = GE \times \frac{1}{2} CD = IL \times \frac{1}{2} CD$. W. W. D.

PROPOSITION XXXVIII.

*The versed Sine of any Arc, drawn into half the
Radius, is equal to the Square of the Sine of one
half of the said Arc.*

THE Triangles CBM, DEB , are equiangular,
since the Angles at M and E are right Angles,
and the Angle at B is common. Wherefore $EB : BD$
 $:: BM : BC$. And then will $EB \times BC = BM \times BD$;
and $EB \times \frac{1}{2} BC = BM \times \frac{1}{2} BD = BM^2$. W. W. D.

PROPOSITION XXXIX.

In any spherical Triangle ABC, whose Legs containing the Angle B are BC, AB, and Base subtending that Angle AC; if the Arc AM be taken = Difference of the Legs = $BC - AB$: Then shall the Rectangle under the Sines of the Legs BC, BA, be to the Square of the Radius, as the Rectangle under the Sine of the Arc $\frac{AC + AM}{2}$

and the Sine of the Arc $\frac{AC - AM}{2}$ is to the Square of the Sine of one Half of the Angle B. Vid. Fig. to Prop. 36.

BEcause the Rectangle under the Sines of the Legs AB, BC, is to the Square of Radius, as IL is to the versed Sine of the Angle B, or as $R \times IL$ to $\frac{1}{2} R$ drawn in the versed Sine of the Angle B (by Prop. 36. of this). And since $\frac{1}{2} R \times IL =$ Rectangle under the Sines of the Arcs $\frac{AC + AM}{2}$ and $\frac{AC - AM}{2}$ (by Prop. 37. of this). And also $\frac{1}{2} R$ drawn into the versed Sine of the Angle B is equal to the Square of the Sine of one Half of the Angle B. Therefore the Rectangle under the Sines of the Sides to the Square of Radius, shall be as the Rectangle under the Sines of the Arcs $\frac{AC + AM}{2}$ and $\frac{AC - AM}{2}$, is to the Square of the Sine of one Half the Angle B. W. W. D.

The Twelve Cases of oblique-angled spherical Triangles are as follow.

	Given	Sought	Make as	In the Original the Proportion was thus;
1	Angles B, D, and BC.	Angle C.	$R : \text{Cof. } BC :: T, B : \text{Cot. } BCA$ (by Prop. 30. of this) : Also $\text{Cof. } B : S, BCA :: \text{Cof. } D : S, DCA$ (by 31 of this). Wherefore the Sum of the Angles BCA, DCA, if the Perpendicular falls within the Triangle, or the Difference, if it falls without, will be = BCD. Whether the Perpendicular falls within or without the Triangle, may be known from the Affection of the Angles B and D (by 22 of this) ; which Admonition ought to be observ'd in the following Solutions.	$\text{Cof. } BC :$ $R :: T B :$ $\text{Cot. } BCA.$
2	Angles B, BCD, and the Side BC.	Angle D.	$R : \text{Cof. } BC :: T, B : \text{Cot. } BCA$ (Prop. 30. of this). And $S, BCA : S, DCA :: \text{Cof. } B : \text{Cof. } D$ (by Prop. 31.) If BCA be less than BCD, the Angle D shall be of the same Affection with the Angle B. If BCA be greater than the Angle BCD, then the Angles B and D shall be of a different Affection, by the Converse of Prop. 22.	This Proportion in the Original was as in the foregoing. The Species of the Angle BCA may be known by Prop. 18. and 19.
3	The Sides BC, CD, and the Angle B.	The Side BD.	$R : \text{Cof. } B :: T, BC : T, BA$ (by 28. of this). And $\text{Cof. } BC :: \text{Cof. } BA :: \text{Cof. } DC : \text{Cof. } DA$ (by 32. of this). The Sum or Difference of BA and DA, according as the Perpendicular falls within or without the Triangle, is equal to BD ; which cannot be known, unless the Species of the Angle D be first known.	

	Given	Sought	Make as
4	The Sides BC, DB, and the Angle B.	The Side CD.	$R : \text{Cof. } B :: T, BC : T, BA$ (by 28. of this). And $\text{Cof. } BA : \text{Cof. } BC :: \text{Cof. } DA : \text{Cof. } DC$ (by <i>Prop.</i> 32. of this). According as DA is similar or dissimilar to CA , or to the Angle BDC , so shall DC be lesser or greater than a Quadrant (by 19. and 20. of this).
5	Angle B, D, and the Side BC.	The Side BD.	$R : \text{Cof. } B :: T, BC : T, BA$ (by 28. of this). And $T, D : T, B :: S, BA : S, DA$ (by 33. of this). The Sum or Difference of which BA and DA is $= BD$.
6	The Sides BC, BD, and the Angle B.	Angle D.	$R : \text{Cof. } B :: T, BC : T, BA$ (by <i>Prop.</i> 28. of this). And $S, DA : S, BA :: T, B : T, D$ (by 33. of this). According as BD is greater or lesser than BA , the Angle D shall be similar or dissimilar to the Angle B (by 22. of this).
7	The Sides BC, DC, and the Angle B.	Angle C.	$\text{Cof. } BC : R :: \text{Cot. } B : T, BCA$, (by 30. of this). And $T, DC : T, BC :: \text{Cof. } BCA : \text{Cof. } DCA$ (by 34. of this). The Sum or Difference of the Angles BCA , DCA , according as the Perpendicular falls within or without the Triangle, is equal to the Angle BCD .
8	The Angles BCD, and B, and the Side BC.	The Side DC.	$\text{Cof. } BC : R :: \text{Cot. } B : T, BCA$ (by 30. of this). Also, $\text{Cof. } DCA : \text{Cof. } BCA :: T, BC : T, DC$ (by 34. of this). If the Angle DCA be similar to the Angle B , (that is, if AD be similar to CA) then DC shall be less than a Quadrant. If the Angles DCA and B be dissimilar, then DC shall be greater than a Quadrant, which follows (from <i>Prop.</i> 18. 19. and 20. of this).

Given

Given	Sought	Make as
9 The Sides BC, DC, and the Angle B.	The Angle D.	$S, CD : S, B :: S, BC : S, D$; which is ambiguous. The Ana- logy follows from <i>Prop.</i> 35. of this.
10 The Angles B, D, and the Side BC.	The Side DC.	$S, D : S, BC :: S, B : S, DC$; which Side is ambiguous.
11 All the Sides AB, BC, CA. <i>Vid.</i> <i>Fig.</i> <i>Prop.</i> 36.	The Angle B.	As the Rectangle under the Sines of the Legs AB, BC : the Square of Radius :: the Rectangle under the Sines of the Arcs $\frac{AC+AM}{2}$ and $\frac{AC-AM}{2}$: the Square of the Sine of $\frac{1}{2}$ the Angle B (by <i>Prop.</i> 39.).
12 All the Angles G, H, D. <i>Vid.</i> <i>Fig.</i> <i>Prop.</i> 14.	The Side GD.	In the Triangle XNM, the Arc MN is the Complement of the Angle GHD to a Semicircle. XM is the Complement of the Angle G, and XN the Com- plement of the Angle D. And the Angle X, the Complement of the Side GD to a Semicircle. Wherefore, if the Angles be changed into Sines, and the Sides into Angles, the Operation will be the same as in <i>Case</i> II. of this; since Arcs and their Com- plements to Semicircles have the same Sines.

The following REMARK by
SAMUEL CUNN.

THAT this is true but in a particular Case, viz. when two of the Angles of the Triangle are right ones, and two of the Sides Quadrants, may be thus demonstrated: For, if possible, let some Triangle RST, *Fig. to Prop. 14th*, be such, that its Sides RS, ST, TR, be equal to the Measures of GHD, HGD, GDH, the Angles of a Triangle GHD; and, also, that the Measures of RST, STR, TRS, the Angles of the Triangle RST, be equal to GH, GD, HD, the Sides of the Triangle GHD. And produce MX, MN, two Sides of the supplemental Triangle, to Semicircles, and they will meet somewhere, suppose at E; and there will be constructed thereby the Triangle NEX, of which XE (the Supplement of XM, which, by the 14th *Prop.* was the Supplement of the Measure of the Angle HGD) is equal to the Measure itself of the same Angle HGD: And, in like manner, NE, the Supplement of NM, which, by the 14th *Prop.* was the Supplement of the Measure of the Angle GHD, is equal to the Measure itself of the same Angle GHD. But the third Side XN is not the Measure of the third Angle GDH, but its Supplement, by the 14th *Prop.* Moreover, of the Angle EXN, (whose Supplement is NXM) the Measure, by the 14th *Prop.* is equal to GD; and of the Angle XNE, (whose Supplement is MNX) the Measure, by the 14th *Prop.* is equal to HD. But of the third NEX, (which is equal to NMX) the Measure is not equal to GH, but its Supplement.

Now make $NV = RT = BK$, the Measure of the Angle GDH, and draw the great Circle EV. And since RS, by Supposition, is equal to the Measure of the Angle GHD, which is equal to EN; and since the Measure of the Angle SRT is, by Supposition, equal to DH, which is also equal to the Measure of the Angle XNE; the Angle XNE is equal to the Angle R. Then, consequently, by the

4th Prop. the Triangles SRT , ENV , will have the Base ST , equal to the Base EV ; the Angle T , to the Angle NVE , and the Angle S , to the Angle NEV . But ST , (which is equal to EV) by Supposition, is equal to the Measure of the Angle HGD ; to which Measure XE is also equal. Therefore EV is equal to XE ; and, consequently, by the 7th Prop. the Angle EVX is equal to the Angle EXV ; and the Angle EXV (whose Measure, as hath been shewn above, is equal to GD) is equal to the Angle T , (or NVE) since, by Supposition, the Measure of this is also equal to GD . Therefore the Angle EVX is equal to the Angle ENV , and so both right ones; and, consequently, EXV a right one also. Therefore, by the 2^d Cor. to the 2^d Prop. EV and EX are both Quadrants.

But if EV be a Quadrant, and at right Angles to NX , then E , by 2^d Prop. and its Coroll. is the Pole of NX ; and so EN a Quadrant also, and the Angle ENV a right one. Therefore, if the Sides of a Triangle (NEV , or its Equal) RST are equal to the Measures of the Angles of some other Triangle GHD , and the Measures of the Angles of the former, equal to the Sides of the latter; two Sides of such a Triangle RST , or GHD , must be Quadrants, and two Angles of each right ones.

Therefore, if a Triangle RST be constructed, whose Sides are equal to the Measure of the Angles of another Triangle GHD ; the Measures of the Angles of the Triangle RST shall not be equal to the Sides of the Triangle GHD , unless in the one Case before-mention'd. Therefore the Measures of the Angles of the Triangle GHD , us'd as the Sides of a Triangle in the 11th Case, will not give us a Side of GHD , but the Measure of an Angle of the Triangle RST , unless in the one aforementioned Case; which was to be demonstrated.

But to find a Side GD of a spherical Triangle GHD , whose Angles are all given, produce MN , that Side of the supplemental Triangle which is equal to the Supplement of the Measure of GHD , the Angle opposite to the Side sought, and MX , either of the other Sides, till they meet as in E . And there, as hath been before shewn, the Sides EX , EN , of the Triangle

angle EXN , are exactly equal to the Measures of the Angles HGD , GHD , of the Triangle GHD ; and of the Angles EXN , ENX , of the Triangle EXN , the Measures are equal to GD , HD . But the Side XN is equal to the Supplement of the Measure of the Angle GDH . And of the Angle XEN , the Measure is equal to the Supplement of GH .

Therefore the SOLUTION is thus :

Change one of the Angles GDH , adjacent to the Side sought, into its Supplement; and then work with the Measures of the Angles as tho' they were Sides; and the Result will be GD , the Side sought.

The preceding Fault, as well as the Omissions hereafter mention'd, are not peculiar to our Author; but may be found in Dr. *Harris*, Mr. *Caswell*, Mr. *Heynes*, and many other Trigonometrical Writers.

In the Solution of our 8th and 9th Cases, they have told us, that the *Quæsitæ* are ambiguous; which sometimes, indeed, is true, but sometimes also false: Therefore, as I conceive it, they ought to have laid down Rules, by Help of which we might discover when the *Quæsitæ* are ambiguous, and when not.

This Oversight may be corrected by the following Directions; wherein, because every Sine corresponds to two Arches, to one less than a Quadrant, and to another, which is the Supplement of the former to a Semicircle, (a true Distinction of which of these are to be us'd, being necessary to be known, before a proper Solution can be given to such Problems as these are) I shall beg Leave, for Brevity-sake, to call the lesser Arch the Acute Value, and the greater the Obtuse; whether the Sine be of an Angle, or a Side.

In the tenth Case, there are given two Angles B, D , and BC , a Side opposite to one of those Angles D , to find DC the Side opposite to the other.

TO the acute Value of DC , and also to its obtuse one, add BC ; and if each of these Sums are greater

$\left\{ \begin{smallmatrix} \text{greater} \\ \text{less} \end{smallmatrix} \right\}$ than a Semicircle, when the Sum of the Angles B, D, is $\left\{ \begin{smallmatrix} \text{greater} \\ \text{less} \end{smallmatrix} \right\}$ than two right Angles; both the Values of DC may be admitted, and then is ambiguous: But when only one of those Sums is $\left\{ \begin{smallmatrix} \text{greater} \\ \text{less} \end{smallmatrix} \right\}$ than a Semicircle, only one Value of DC can be true, viz. the $\left\{ \begin{smallmatrix} \text{obtuse} \\ \text{acute} \end{smallmatrix} \right\}$ one; and then is not ambiguous.

In the ninth Case, there are given two Sides BC, DC, and one Angle B, opposite to DC, one of those Sides, to find D the Angle opposite to the other.

TO the acute Value of D, and also to its obtuse Value, add B; and if each of these Sums is $\left\{ \begin{smallmatrix} \text{greater} \\ \text{less} \end{smallmatrix} \right\}$ than two right Angles, when the Sum of the Sides is $\left\{ \begin{smallmatrix} \text{greater} \\ \text{less} \end{smallmatrix} \right\}$ than a Semicircle, both the Values of D may be admitted, and consequently D is ambiguous: But when only one of those Sums is $\left\{ \begin{smallmatrix} \text{greater} \\ \text{less} \end{smallmatrix} \right\}$ than two right Angles, only one Value of D is true, viz. the $\left\{ \begin{smallmatrix} \text{obtuse} \\ \text{acute} \end{smallmatrix} \right\}$ one; and then not ambiguous.

Nor are we better us'd in the first Proposition; for tho' it is determin'd by the given Angles, whether the Perpendicular falls within or without the Triangles, yet, in each of those Varieties, the *Quæsitæ* will be sometimes ambiguous, and sometimes not.

In this first Proposition there are given two Angles B, D, and BC, a Side opposite to D, one of them, to find C the third Angle.

1. Let the Perpendicular fall within ; that is, let the given Angles be of the same Species.

TO the acute Value of DCA, and also to its obtuse one, add the Angle BCA; and if each of these Sums is less than two right Angles, then either the acute Value of DCA, or its obtuse one added to BCA, gives a Value of BCD; which, therefore, is ambiguous. And when only one of these Sums is less than two right Angles, the acute Value of DCA, added to BCA, gives the only Value of BCD; which then is not ambiguous, though in both Varieties the Perpendicular fell within.

2. Let the Perpendicular fall without ; that is, let the given Angles be of different Species.

WHEN the obtuse Value of the Angle DCA is less than the Angle BCA, the Angle BCD may be had by subtracting either Value of DCA from BCA; and then BCD is ambiguous. But when the obtuse Value of DCA is not less than BCA, the acute Value of DCA, taken from BCA, gives the single Value of BCD; which, therefore, is not ambiguous; tho' in both Varieties the Perpendicular fell without.

In the fifth Case we lie under the same Misfortune, where there are given, as in the first, the Angles B, D, and the Side BC, to find BD the Side lying between those given Angles.

1. When the Perpendicular falls within ; that is, when the given Angles are of the same Species.

TO the acute Value of DA, and so also to its obtuse one, add BA; and if each of these Sums is less than a Semicircle, then either the acute Value of DA, or its obtuse one, added to BA, gives the Value

Value of BD ; which thence is ambiguous. And when only one of these Sums is less than a Semicircle, the acute Value of DA , added to BA , gives the only Value of BD ; which then is not ambiguous, tho' in both Varieties the Perpendicular fell within.

2. When the Perpendicular falls without; that is, when the given Angles are of different Species.

WHEN the obtuse Value of DA is less than BA , BD will be had by subtracting either Value of DA from BA ; and then BD is ambiguous. But when the obtuse Value of DA is not less than BA , the acute Value of DA , taken from BA , leaves the only Value of BD ; which, therefore, is not ambiguous, tho' in both Varieties the Perpendicular fell without.

In the third, we have the same Omission; where there are given two Sides BC , CD , and B an Angle opposite to CD one of them, to find the third Side BD .

FIRST, we may observe, that the Species of DA is always known; for it is of $\left\{ \begin{array}{l} \text{the same} \\ \text{a different} \end{array} \right\}$ Affection with the Angle B , when DC is $\left\{ \begin{array}{l} \text{less} \\ \text{greater} \end{array} \right\}$ than a Quadrant. And,

If AD be less than AB , and also the Sum of AD and AB less than a Semicircle; then AD , either added to, or subtracted from AB , will give the Value of BD ; which, therefore, is ambiguous.

But if AD be not less than AB , or if their Sum be not less than a Semicircle; then their Sum in the former, and their Difference in the latter Variety, shall give one single Value of BD ; and then is not ambiguous.

The seventh Case much resembles the third; for there are given two Sides BC, CD, and B an Angle, opposite to CD one of them; to find the Angle BCD, lying between those two Sides.

AND here we may observe, that the Species of the Angle DCA is known; for it is of $\left\{ \begin{array}{l} \text{the same} \\ \text{a different} \end{array} \right\}$ Kind with the Angle B, when DC is $\left\{ \begin{array}{l} \text{less} \\ \text{greater} \end{array} \right\}$ than a Quadrant. And,

If DCA be less than BCA, and the Sum of DCA and BCA less than two right Angles; then DCA either added to, or subtracted from BCA, will give the Angle BCD; which, therefore, is ambiguous.

If DCA be not less than BCA, or the Sum of DCA and BCA not less than two right Angles; then their Sum in the former, and their Difference in the latter Variety, shall give the single Value of BCD; which, then, is not ambiguous.

N. B. If any one will be at the Trouble to make a double Calculation for the Side DC, or the Angle D, as taught in the Remarks on the 9th and 10th Cases, they will find the several Varieties in the 1st, 3d, 5th, and 7th, to be as here laid down in these easy Rules.

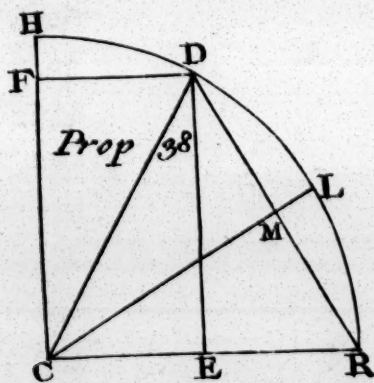
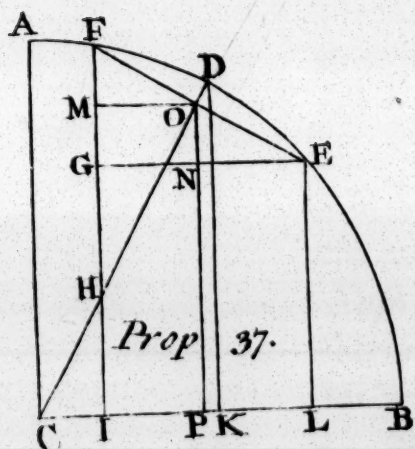
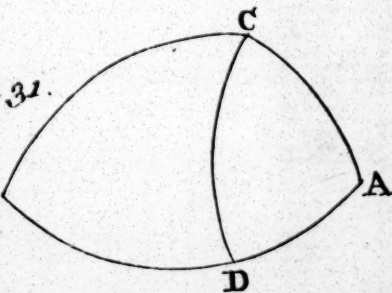
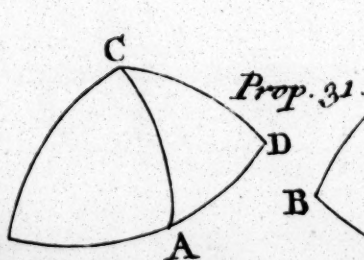
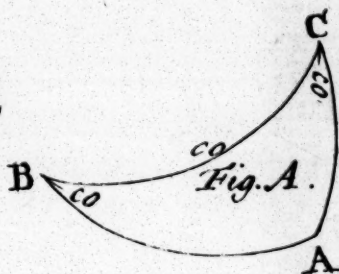
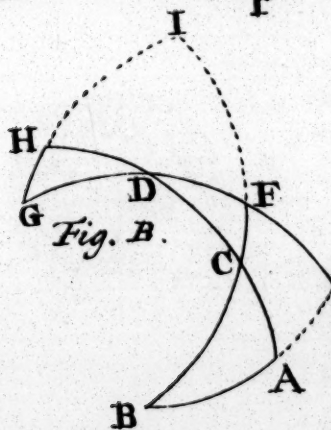
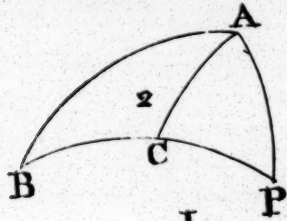
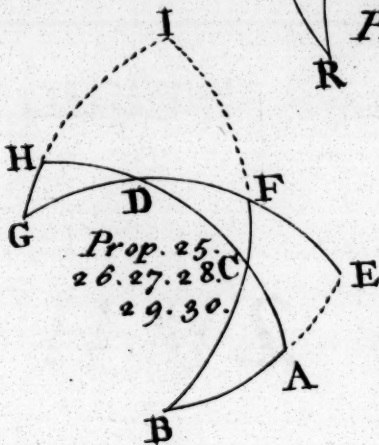
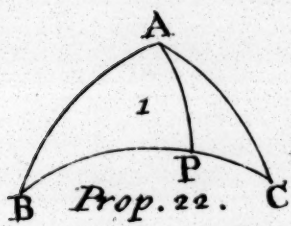
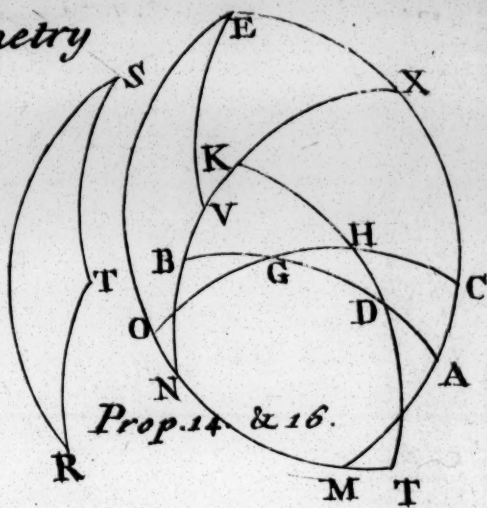
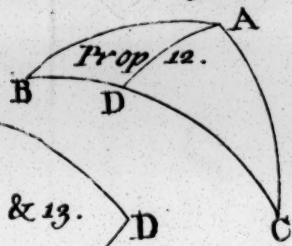
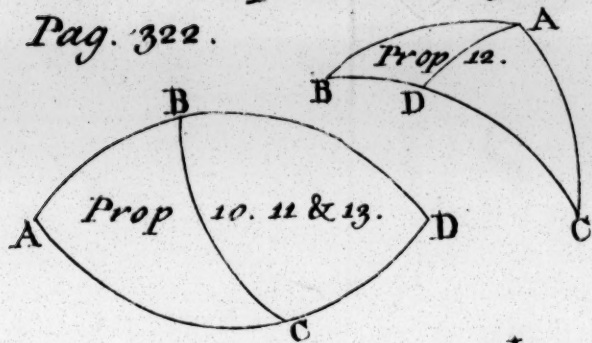
The Truth of these Rules may be easily deduced from the 10th, 13th, 18th, and 22d *Prop. of this*, and the 2d, 8th, and 13th *Examples* following *Prop. 30. of this*.

In our third *Case* of oblique plain Triangles, our Author should have added this:

If AB be less than BC, the Angle A is ambiguous; otherwise, not.

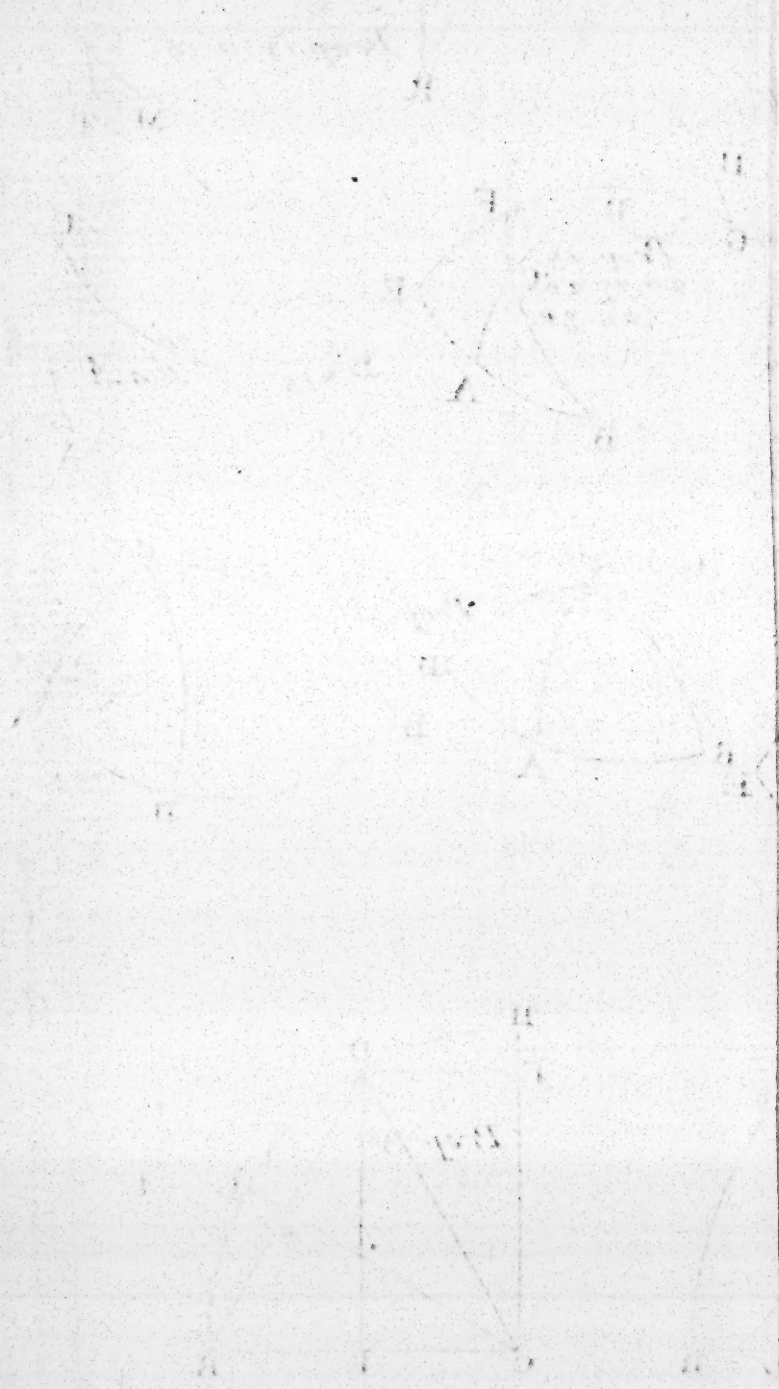
Spherical Trigonometry

Pag. 322.



8

Verzeichnis



A SHORT
T R E A T I S E
O F T H E
Nature *and* Arithmetic
O F
L O G A R I T H M S.

The P R E F A C E.

TH E Mathematics formerly received considerable Advantages ; first, by the Introduction of the Indian Characters, and afterwards by the Invention of Decimal Fractions ; yet has it since reaped at least as much from the Invention of Logarithms, as from both the other Two. The Use of these, every one knows, is of the greatest Extent, and runs through all Parts of Mathematics. By their Means it is that Numbers almost infinite, and such as are otherwise impracticable, are managed with Ease and Expedition. By their Assistance the Mariner steers his Vessel, the Geometrician investigates the Nature of the higher Curves, the Astronomer determines the Places of the Stars, the Philosopher accounts for
Y other

other Phænomena of Nature; and, lastly, the Usurer computes the Interest of his Money.

The Subject of the following Treatise has been cultivated by Mathematicians of the first Rank; some of whom, taking in the whole Doctrine, have indeed written learnedly, but scarcely intelligible to any but Masters. Others, again, accommodating themselves to the Apprehension of Novices, have selected out some of the most easy and obvious Properties of Logarithms, but have left their Nature and more intimate Properties untouch'd. My Design therefore in the following Tract, is to supply what seemed still wanting, viz. to discover and explain the Doctrine of Logarithms, to those who are not yet got beyond the Elements of Algebra and Geometry.

The wonderful Invention of Logarithms we owe to the Lord Napier, who was the first that constructed and published a Canon thereof, at Edinburgh, in the Year 1614. This was very graciously received by all Mathematicians, who were immediately sensible of the extreme Usefulness thereof. And tho' it is usual to have various Nations contending for the Glory of any notable Invention, yet Napier is universally allowed the Inventor of Logarithms, and enjoys the whole Honour thereof without any Rival.

*The same Lord Napier afterwards invented another and more commodious Form of Logarithms, which he communicated to Mr. Henry Briggs, Professor of Geometry at Oxford, who was hereby introduced as a Sharer in the completing thereof: But the Lord Napier dying, the whole Business remaining was devolved upon Mr. Briggs, who with prodigious Application, and an uncommon
Dexterity*

Dexterity, compos'd a Logarithmic Canon, agreeable to that new Form for the First twenty Chiliads of Numbers, (or from 1 to 20000) and for Eleven other Chiliads, viz. from 90000 to 101000. For all which Numbers he calculated the Logarithms to fourteen Places of Figures. This Canon was published at London in the Year 1624.

Adrian Vlacq published again this Canon at Gouda in Holland, in the Year 1628. with the intermediate Chiliads before omitted, filled up according to Briggs's Prescriptions; but these Tables are not so useful as Briggs's, because the Logarithms are continued but to 10 Places of Figures.

Mr. Briggs also has calculated the Logarithms of the Sines and Tangents of every Degree, and the Hundredth Parts of Degrees to 15 Places of Figures; and has subjoined to them the natural Sines, Tangents, and Secants, to 15 Places of Figures. The Logarithms of the Sines and Tangents are called artificial Sines and Tangents. These Tables, together with their Construction and Use, were publish'd after Briggs's Death, at London, in the Year 1633. by Henry Gillibrand, and by him called Trigonometria Britannica.

Since then there have been published, in several Places, compendious Tables, wherein the Sines and Tangents, and their Logarithms, consist of but seven Places of Figures, and wherein are only the Logarithms of the Numbers from 1 to 100000, which may be sufficient for most Uses.

The best Disposition of these Tables, in my Opinion, is that first thought of by Nathanael Roe, of Suffolk; and, with some Alterations for the better.

better, followed by Sherwin in his Mathematical Tables, published at London in 1705; wherein are the Logarithms from 1 to 101000, consisting of seven Places of Figures. To which are subjoined the Differences, and proportional Parts, by means of which may be found easily the Logarithms of Numbers to 10000000, observing at the same time, that these Logarithms consist only of seven Places of Figures. Here are also the Sines, Tangents, and Secants, with the Logarithms and Differences for every Degree and Minute of the Quadrant, with some other Tables of Use in practical Mathematics.

O F T H E

Nature *and* Arithmetic

O F

LOGARITHMS.

C H A P. I.

Of the O R I G I N *and* N A T U R E *of*
LOGARITHMS.

AS in Geometry, the Magnitudes of Lines are often defined by Numbers; so likewise, on the other hand, it is sometimes expedient to expound Numbers by Lines, *viz.* by assuming some Line which may represent Unity, and the Double thereof; the Number 2, the Triple 3, the one half, the Fraction $\frac{1}{4}$, and so on. And thus the Genesis and Properties of some certain Numbers are better conceived, and more clearly considered, than can be done by abstract Numbers.

Hence, if any Line a^* be drawn into itself, the Quan- * *Fig. 1.*
tity a^2 produced thereby, is not to be taken as one of two Dimensions, or as a Geometrical Square, whose Side is the Line a , but as a Line that is a third Proportional to some Line taken for Unity, and the Line n . So likewise, if a^2 be multiplied by a , the Product a^3 will not be a Quantity of three Dimensions, or a Geometrical Cube, but a Line that is the fourth Term in a Geometrical Progression, whose first Term is 1, and second a ; for the Terms 1, a , a^2 , a^3 , a^4 , a^5 , a^6 , a^7 , &c. are in the continual Ratio of 1 to a . And the Indices affixed to the Terms shew the Place or
Y 3 Distance

Distance that every Term is from Unity. For Example, a^5 is in the fifth Place from Unity, a^6 is the sixth, or six times more distant from Unity than a , or a^1 , which immediately follows Unity.

If, between the Terms 1 and a , there be put a mean Proportional, which is $\sqrt{a}$, the Index of this will be $\frac{1}{2}$; for its Distance from Unity will be one Half of the Distance of a from Unity; and so $a^{\frac{1}{2}}$ may be written $\sqrt{a}$. And if a mean Proportional be put between a and a^2 , the Index thereof will be $\frac{3}{2}$ or $\frac{1}{2}$, for its Distance will be sesquialteral of the Distance a from Unity.

If there be two mean Proportionals put between 1 and a ; the first of them is the Cube Root of a , whose Index must be $\frac{1}{3}$, for that Term is distant from Unity only by a third Part of the Distance of a from Unity; and so the Cube Root must be expressed by $a^{\frac{1}{3}}$. Hence, the Index of Unity is 0; for Unity is not distant from itself.

The same Series of Quantities, geometrically proportional, may be both Ways continued, as well descending towards the Left Hand, as ascending towards the

Right; for the Terms $\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4}, \frac{1}{a^5}, 1, a, a^2, a^3, a^4, a^5, \&c.$ are all in the same Geometrical Progression. And since the Distance of a from Unity is towards the Right Hand; and positive or $+1$, the Distance equal to that on the contrary Side, *viz.* the Distance of the Term $\frac{1}{a}$, will be Negative or -1 ,

which shall be the Index of the Term $\frac{1}{a}$, for which

may be written a^{-1} . So likewise in the Term a^{-2} the Index -2 shews, that that Term stands in the second Place from Unity towards the Left Hand, and the Terms a^{-2} and $\frac{1}{a^2}$ are of the same Value. Also

a^{-3} is the same as $\frac{1}{a^3}$. For these negative Indices shew, that the Terms belonging to them go from Unity the contrary Way to that by which the Terms, whose Indices are positive, do. These things premised,

If on the Line AN, both Ways indefinitely extended, be taken, AC, CE, EG, GI, IL, on the Right Hand; and also AG, GP, &c. on the Left, all equal to one another; and if, at the Points P, G, A, C, E, G, I, L, be erected to the Right Line AN, the Perpendiculars $\Pi\Sigma$, $\Gamma\Delta$, AB, CD, EF, GH, IK, LM, which let be continually proportional, and represent Numbers, whereof AB is Unity: The Lines AC, AE, AG, AI, AL, — AG, — AP, respectively express the Distances of the Numbers from Unity, or the Place and Order that every Number obtains in the Series of, Geometrical Proportionals, according as it is distant from Unity. So since AG is triple of the Right Line AC, the Number GH shall be in the third Place from Unity, if CD be in the first: So likewise shall LM be in the fifth Place, since $AL = 5 AC$. If the Extremities of the Proportionals, $\Sigma, \Delta, B, D, F, H, K, M$, be joined by Right Lines, the Figure $\Sigma\Pi LM$ will become a Polygon consisting of more or less Sides, according as there are more or less Terms in the Progression.

If the Parts AC, CE, EG, GI, IL, be bisected in the Points c, e, i, g, l , and there be again raised the Perpendicular cd, ef, gh, ik, lm , which are mean Proportionals between AB, CD; CD, EF; EF, GH; GH, IK; IK, LM; then there will arise a new Series of Proportionals, whose Terms, beginning from that which immediately follows Unity, are double of those in the first Series, and the Differences of the Terms are become less, and approach nearer to a Ratio of Equality than before. Likewise in this new Series, the Right Lines AL, AC, express the Distances of the Terms LM, CD, from Unity; viz. Since AL is ten times greater than Ac, LM shall be the tenth Term of the Series from Unity: And because Ae is three times greater than Ac, ef will be the third Term of the Series, if cd be the first; and there shall be two mean Proportionals between AB and ef; and between AB and LM there will be nine mean Proportionals.

And if the Extremities of the Lines Bd, Df, Fh, H, &c. be joined by Right Lines, there will be a new Polygon made, consisting of more, but shorter Sides than the last.

If, again, the Distances $A c, c C, C e, e E, \&c.$ be supposed to be bisected, and mean Proportionals between every two of the Terms be conceived to be put at those middle Distances; then there will arise another Series of Proportionals, containing double the Number of Terms from Unity than the former does; but the Differences of the Terms will be less, and if the Extremities of the Terms be joined, the Number of the Sides of the Polygon will be augmented according to the Number of Terms; and the Sides thereof will be lesser, because of the Diminution of the Distances of the Terms from each other.

Now, in this new Series, the Distances $A L, A C, \&c.$ will determine the Orders or Places of the Terms; *viz.* If $A L$ be five times greater than $A C$, and $C D$ be the fourth Term of the Series from Unity, then $L M$ will be the twentieth Term from Unity.

If in this manner mean Proportionals be continually placed between every two Terms, the Number of Terms at last will be made so great, as also the Number of the Sides of the Polygon, as to be greater than any given Number, or to be infinite; and every Side of the Polygon so lessened, as to become less than any given Right Line; and consequently the Polygon will be changed into a curve-lined Figure; for any curve-lined Figure may be conceived as a Polygon, whose Sides are infinitely small, and infinite in Number.

A Curve described after this manner is called *Logarithmical*; in which, if Numbers be represented by Right Lines standing at Right Angles to the Axis $A N$, the Portion of the Axis intercepted between any Number and Unity, shews the Place or Order that that Number obtains in the Series of Geometrical Proportionals, distant from each other by equal Intervals. For Example, if $A L$ be five times greater than $A C$, and there are a thousand Terms in continual Proportion from Unity to $L M$; then will there be two hundred Terms of the same Series from Unity to $C D$, or $C D$ shall be the two hundredth Term of the Series from Unity; and let the Number of Terms from $A B$ to $L M$ be supposed what it will, then the Number of Terms from $A B$ to $C D$ will be one fifth Part of that Number.

The

The Logarithmical Curve may also be conceived to be described by Two Motions, one of which is equable, and the other accelerated, or retarded, according to a given Ratio. For Example, if the Right Line AB moves uniformly along the Line AN , so that the End thereof describes equal Spaces in equal Times; and, in the mean time, the said Line AB so increases, that the Increments thereof, generated in equal Times, be proportional to the whole increasing Line; that is, if AB , in going forward to cd , be increased by the Increment od , and in an equal Time when it is come to CD , the Increment thereof is Dp , and Dp to dc is as do is to AB , that is, if the Increments generated in equal times are always proportional to the Wholes; or, if the Line AB , moving the contrary Way, diminishes in a constant Ratio, so that while it goes through the equal Spaces, the Decrements $AB - \Gamma\Delta$, $\Gamma\Delta - \Pi\Sigma$, are Proportionals to AB , $\Gamma\Delta$; then the End of the Line, increasing or decreasing in the said manner, describes the Logarithmical Curve: For since $AB : do :: dc : Dp :: DC : fq$, it shall be (by Composition of Ratio) as $AB : dc :: dc : DC :: DC : fe$, and so on.

By these two Motions, *viz.* the one equable, and the other proportionally accelerated or retarded, the Lord *Napier* laid down the Origin of Logarithms, and called the Logarithm of the Sine of any Arc, *That Number which nearest defines a Line that equally increases, while, in the mean time, the Line expressing the whole Sine proportionally decreases to that Sine.*

It is manifest from this Description of the Logarithmic Curve, that all Numbers at equal Distances are continually proportional. It is also plain, that if there be four Numbers AB, CD, IK, LM , such, that the Distance between the first and second be equal to the Distance between the third and the fourth: Let the Distance from the second to the third be what it will, these Numbers will be proportional. For, because the Distances AC, IL , are equal, AB shall be to the Increment Ds , as IK is to the Increment MT . Wherefore (by Composition) $AB : DC :: IK : ML$. And contrariwise, if four Numbers be proportional, the Distance between the first and the second shall be

be equal to the Distance between the third and the fourth.

The Distance between any two Numbers is called the Logarithm of the Ratio of those Numbers, and indeed doth not measure the Ratio itself, but the Number of Terms in a given Series of Geometrical Proportionals proceeding from one Number to another; and defines the Number of equal Ratio's by the Composition whereof the Ratio's of Numbers are known.

If the Distance between any two Numbers be double to the Distance between two other Numbers, then the Ratio of the two former Numbers shall be the Duplicate of the Ratio of the two latter. For let the Distance IL between the Numbers IK , LM , be double to the Distance Ac , between the Numbers AB , cd ; and since IL is bisected in l , we have $Ac = Il = lL$; and the Ratio of IK to lm is equal to the Ratio of AB to cd ; and so the Ratio of IK to LM , the Duplicate of the Ratio of IK to lm , (by *Def. 10. El. 5.*) shall be the Duplicate of the Ratio of AB to cd .

In like manner, if the Distance EL be triple of the Distance AC , then will the Ratio of EF to LM , be triplicate of the Ratio of AB to CD : For, because the Distance is triple, there shall be three times more Proportionals from EF to LM , than there are Terms of the same Ratio from AB to CD ; and the Ratio of EF to LM , as also of AB to CD , is compounded of the equal intermediate Ratio's (by *Def. 5. El. 6.*). And so the Ratio of EF to LM , compounded of three times a greater Number of Ratio's, shall be triplicate of the Ratio of AB to CD . So likewise, if the Distance GL be quadruple of the Distance Ac , then shall the Ratio of GH to LM , be quadruplicate of the Ratio of AB to cd .

The Logarithm of any Number is the Logarithm of the Ratio of Unity to that Number, or it is the Distance between Unity and that Number. And so Logarithms express the Power, Place, or Order, which every Number, in a Series of Geometrical Progressionals, obtains from Unity. For Example, if there be 10000000 proportional Numbers from Unity to the Number 10, that is, if the Number 10 be in the 10000000th Place from Unity; then it will be found,

by

by Computation, that in the same Series from Unity, to 2, there are 3010300 proportional Terms; that is, the Number 2 will stand in the 3010300th Place. In like manner, from Unity to 3, there will be found 4771213 proportional Terms, which Number defines the Place of the Number 3. The Numbers 10000000, 3010300, 4771213, shall be the Logarithms of the Numbers 10, 2, and 3.

If the first Term of the Series from Unity be called y , the second Term will be y^2 , the third y^3 , &c. And since the Number 10 is the 10,000,000th Term of the Series, then will $y^{10000000} = 10$. Also $y^{3010300} = 2$. Also $y^{4771213} = 3$; and so on.

Wherefore all Numbers shall be some Powers of that Number which is the first from Unity; and the Indices of the Powers are the Logarithms of the Numbers.

Since Logarithms are the Distances of Numbers from Unity, as has been shewn, the Logarithm of Unity shall be 0; for Unity is not distant from itself, but the Logarithms of Fractions are negative, or descending below nothing; for they go on the contrary Way. And so if Numbers, increasing proportionally from Unity, have positive Logarithms, or such as are affected with the Sign +; then Fractions or Numbers, in like manner decreasing, will have negative Logarithms, or such as are affected with the Sign -; which is true when Logarithms are considered as the Distances of Numbers from Unity.

But if Logarithms take their Beginning not from an integral Unit, but from an Unit that is in some Place of decimal Fractions; for Example, from the Fraction $\frac{1}{1000000000}$; then all Fractions greater than this, will have positive Logarithms; and those that are less, will have negative Logarithms. But more shall be said of this hereafter.

Since in the Numbers continually proportional, DC, EF, GH, IK, &c. the Distances CE, EG, GI, &c. are equal, the Logarithms AC, AE, AG, AI, &c. of those Numbers, shall be equidifferent, or the Differences of them shall be equal: And so the Logarithms of proportional Numbers are all in an Arithmetical Progression; and from hence proceeds that common Definition of Logarithms, that Logarithms

rithms are Numbers which, being adjoined to Proportions, have equal Differences.

In the first Kind of Logarithms that *Napier* published, the first Term of the continual Proportionals was placed only so far distant from Unity, as that Term exceeded Unity. For Example, if $v n$ be the first Term of the Series from Unity $A B$, the Logarithm thereof, or the Distance $A n$, or $B y$, was, according to him, equal to $v y$, or the Increment of the Number above Unity. As suppose $v n$ be 1,0000001, he placed 0,0000001 for its Logarithm $A n$; and from hence, by Computation, the Number 10 shall be the 2302585th Term of the Series, which Number therefore is the Logarithm of 10 in this Form of Logarithms, and expresses its Distance from Unity in such Parts whereof $v y$ or $A n$ is one.

But this Position is intirely at Pleasure; for the Distance of the first Term may have any given Ratio to the Excess thereof above Unity; and according to that various Ratio, (which may be supposed at Pleasure) that is between $v y$ and $B y$, the Increment of the first Term above Unity, and the Distance of the same from Unity, there will be produced different Forms of Logarithms.

This first Kind of Logarithms was afterwards changed by *Napier*, into another more convenient one, wherein he put the Number 10 not as the 2302585th Term of the Series, but the 1000,000th; and in this Form of Logarithms, the first Increment $v y$ shall be to the Distance $B y$, or $A n$, as Unity, or $A B$, is to the Decimal Fraction 0,4342994, which therefore expresses the Length of the Subtangent $A T$, *Fig. 4.*

After *Napier's* Death, the excellent Mr. *Henry Briggs*, by great Pains, made and published Tables of Logarithms according to this Form. Now since in these Tables the Logarithm of 10, or the Distance thereof from Unity, is 1,0000000, and 1, 10, 100, 1000, 10000, &c. are continual Proportionals, they shall be equidistant. Wherefore the Logarithm of the Numb. 100 shall be 2,0000000; of 1000, 3,0000000; and the Logarithm of 10000 shall be 4,0000000; and so on.

Hence the Logarithms of all Numbers between 1 and 10, must begin with 0, or 0 must stand in the first

first Place to the Left-hand ; for they are lesser than the Logarithm of the Number 10, whose Beginning is Unity ; and the Logarithms of the Numbers between 10 and 100 begin with Unity ; for they are greater than 1,0000000, and less than 2,0000000. Also the Logarithms between 100 and 1000, begin with 2 ; for they are greater than the Logarithm of 100, which begins with 2, and less than the Logarithm of 1000, that begins with 3. In the same manner it is demonstrated, that the first Figure to the Left-hand of the Logarithms between 1000 and 10000, must be 3 ; and the first Figure to the Left-hand of the Logarithms between 10000 and 100000, will be 4 ; and so on.

The first Figure of every Logarithm to the Left-hand, is called the Characteristic or Index, because it shews the highest or most remote Place of the Number from the Place of Units. For Example, if the Index of a Logarithm be 1, then the highest or most remote Place from Unity of the correspondent Number to the Left-hand, will be the Place of Tens. If the Index be 2, the most remote Figure of the correspondent Number shall be in the second Place from Unity ; that is, it shall be in the Place of Hundredths ; and if the Index of a Logarithm be 3, the last Figure of the Number answering to it, shall be in the Place of Thousandths. The Logarithms of all Numbers that are in decuple or subdecuple Progression, only differ in their Characteristics, or Indices, they being written in all other Places with the same Figures. For Example, the Logarithms of the Numbers 17, 170, 1700, 17000, are the same, unless in their Indices ; for since 1 is to 17, as 10 to 170, and as 100 to 1700, and as 1000 to 17000 ; therefore the Distances between 1 and 17, between 10 and 170, between 100 and 1700, and between 1000 and 17000, shall be all equal. And so, since the Distance between 1 and 17, or the Logarithm of the Number 17, is 1.2304489, the Logarithm of the Numb. 170 will be = 2.2304489, and the Logarithm of the Number 1700 shall be 3.2304489, because the Logarithm of the Number 100 = 2.0000000. In like manner, since the Logarithm of the Number 1000 = 3.0000000, the Logarithm of the Number 17000 shall 4.2304489.

Of LOGARITHMS.

So also the Numbers, 6748. 674, 8. 67, 48. 6, 748. 0, 6748. 0, 06748, are continual Proportionals in the Ratio of 10 to 1 ; and so

their Distances from each	6 7 4 8	3,8291751
other shall be equal to the	6 7 4, 8	2,8291751
Distance or Logarithm of	6 7, 4 8	1,8291751
the Number 10, or equal	6, 7 4 8	0,8291751
to 1,0000000. And so,	0,6 7 4 8	-1,8292751
since the Logarithm of 0,0	0 6 7 4 8	-2,8291751

the Number 6748 is

3,8291751, the Logarithms of the other Numbers shall be as in the Margin ; where you may observe, that the Indices of the last two Logarithms are only negative, and the other Figures positive ; and so, when those other Figures are to be added, the Indices must be subtracted, and contrariwise.

C H A P. II.

Of the Arithmetic of Logarithms in whole Numbers, or whole Numbers adjoined to Decimal Fractions. Fig. 2.

BEcause, in Multiplication, Unity is to the Multiplier, as the Multiplicand is to the Product, the Distance between Unity and the Multiplier, shall be equal to the Distance between the Multiplicand and the Product. If therefore the Number GH be to be multiplied by the Number EF, the Distance between GH and the Product must be equal to the Distance AE, or to the Logarithm of the Multiplier; and so, if GL be taken equal to AE, the Number LM shall be the Product ; that is, if the Logarithm of the Multiplicand AG be added to the Logarithm of the Multiplier AE, the Sum shall be the Logarithm of the Product.

In Division, the Divisor is to Unity, as the Dividend is to the Quotient ; and so the Distance between the Divisor and Unity shall be equal to the Distance between the Dividend and the Quotient. So if LM be to be divided by EF, the Distance EA shall be equal to the Distance between LM and the Quotient; and

and so, if LG be taken equal to EA , the Quotient will be at G ; that is, if from AL , the Logarithm of the Dividend, by taken GL , or AE , the Logarithm of the Divisor, there will remain AG , the Logarithm of the Quotient.

And from hence it appears, that whatsoever Operations in common Arithmetic are performed by multiplying or dividing of great Numbers, may be much easier, and more expediently, done by the Addition or Subtraction of Logarithms.

For Example, Let the Number 7589 be to be multiplied by 6757. Now, if the Logarithms of those Numbers be added together, as in the Margin, their Sum will be the Logarithm of the Product, whose Index 7

Log. 3.	8801846
Log. 3.	8297539
Log. 7.	<u>7099385</u>

shews, that there are seven Places of Figures, besides Unity, in the Product; and in seeking this Logarithm in Tables, or the nearest equal to it, I find that the Number answering thereto, which is lesser than the Product, is 51278000; and the Number greater than the Product is 51279000; and if the adjoined Differences, and proportional Parts, be taken, the Numbers that must be added to the Place of Hundreds and Tens in the Product are 87; and that which must be added in the Place of Unity, will necessarily be 3, since seven times 9=63; and so the true Product shall be 51278873. If the Index of the Logarithm had been 8 or 9, then the Numbers to be added in the Place of Hundredths or Tenths could not be had from those Tables of Logarithms which consist but of 7 Places of Figures, besides the Characteristic; and so in this Case the *Vlacquian* or *Briggian* Tables should be used; in the former of which, the Logarithms are all to ten Places of Figures, and in the latter to fourteen.

If the Number 78956 be to be divided by 278, by subtracting the Logarithm of the Divisor from the Logarithm of the Dividend, the Logarithm of the Quotient will be had. And to this Logarithm, the Number 282, 719 answers; which therefore shall be the Quotient.

Log. 4.	8954004
Log. 2.	4440448
Log. 2.	<u>4513550</u>

Because Unity, any assumed Number, the Square thereof, the Cube, the Biquadrate, &c. are all continual

tinual Proportionals, their Distances from each other shall be equal to one another. And so it is manifest, that the Distance of the Square from Unity, is double of the Distance of its Root from the same: Also the Distance of the Cube is triple of the Distance of its Root; and the Distance of the Biquadrate is quadruple of the Distance of its Root from Unity, &c. And so, if the Logarithm of any Number be doubled, we shall have the Logarithm of its Square; if it be tripled, we shall have the Logarithm of its Cube; and if it be quadrupled, the Logarithm of its Biquadrate. And contrariwise, if the Logarithm of any Number be bisected, we shall have the Logarithm of the square Root thereof: Moreover, a third Part of the said Logarithm will be the Logarithm of the Cube Root of the Number; and a fourth Part, the Logarithm of the Biquadrate Root of that Number.

Hence, the Extractions of all Roots are easily performed, by dividing a Logarithm into as many Parts as there are Units in the Index of the Power. So if you want the Square Root of 5, the half of 0,6989700 must be taken, and then that half 0.3494850 will be the Logarithm of the Square Root of 5, or the Logarithm of $\sqrt{5}$, to which the Number 2.23606 nearly answers.

C H A P. III.

Of the Arithmetic of Logarithms, when the Numbers are Fractions. Fig. 3.

WHEN Fractions are to be worked by Logarithms, it is necessary, for avoiding the Trouble of adding one Part of a Logarithm, and subtracting the other, that Logarithms do not begin from an integral Unit, but from some Unit that is the tenth or hundredth Place of Decimal Fractions: For Example, let P O be $\frac{1}{10000000000}$, and from this let the Logarithm begin. Now this Fraction is ten times more distant from Unity to the Left-hand, than the Number 10 is distant therefrom to the Right; for there are 10 proportional Terms in the Ratio of 10 to 1, from Unity to P O. And so, if A B be Unity,

the

the Logarithm thereof, according to this Supposition, will not be 0, but OA will be $\equiv 10.0000000$; for the Distance of any Tenth from Unity is 1.0000000 , whence the Distance of the Number 10 from PO will be 11.0000000 . Also, the Distance of the Number 100 from PO, or its Logarithm, beginning from PO, shall be 12.0000000 ; and the Logarithm of 1000, or the Distance from PO, will be 13.0000000 . And thus, the Indices of all Logarithms are augmented by the Number 10; and those Fractions whose Indices are -1 , or -2 , or -3 , &c. are now made 9, 8, or 7, &c.

But if Logarithms begin from the Place of a Fraction, whose Numerator is Unity, and Denominator Unity with 100 Cyphers added to it, (which they must do when Fractions occur that are less than PO) then that Fraction will be 100 times more distant from Unity, than 10 is distant from it; and so the Logarithm of Unity will have 100 for the Index thereof. And the Logarithm of any Tens will have 101 for the Index, that of any Hundreds 102, and so on; all the Indices being augmented by the Number 100.

The Logarithms of all Fractions that are greater than PO (whereat they begin) will be positive. And since the Numbers 10, 1, $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c. are in a continued Geometrical Progression, they will be equally distant from each other; and accordingly their Logarithms will be equidifferent: And so, when the Logarithm of 10 is 11.0000000 , and the Logarithm of Unity is 10.0000000 , and the Logarithm of the Fraction $\frac{1}{10}$ will be 9.0000000 , and the Logarithm of the Fraction $\frac{1}{100}$ will be 8.0000000 ; and, in like manner, the Index of the Logarithm of $\frac{1}{1000}$ will be 7. Also, for the same Reason, if the Index of the Logarithm of Unity be 100, and of 10 be 101, then will the Index of the Logarithm of the Fraction $\frac{1}{10}$ be 99, and the Index of the Logarithm of $\frac{1}{100}$ will be 98, and the Index of the Logarithm of the Fraction $\frac{1}{1000}$ shall be 97, &c. And these Indices shew in what Place from Unity, the first Figure of the Fraction, not being a Cypher, must be put. For Example, if the Index be 4, the Distance thereof from the Index of Unity, (which is 10) viz. 6, shews that the first significative Figure of the Decimal, is in the sixth Place from Unity; and therefore five Cyphers are to be prefixed

Z

thereto

thereto towards the Left Hand. So, also, if the Index of Unity be 100, and the Index of the Fraction be 80, the first Figure thereof shall be in the 20th Place from Unity, and 19 Cyphers are to be prefix'd thereto.

Now, let it be requir'd to multiply the Fraction GH by the Fraction DC. Because Unity is to the Multiplier, as the Multiplicand is to the Product; the Distance between Unity and the Multiplier shall be equal to the Distance between the Multiplicand and the Product. Therefore, if there be taken $GI = AC$, the Product IK shall be at I. And, accordingly, if from OG, the Logarithm of the Multiplicand, there be taken GI or AC, there will remain OI, the Logarithm of the Product. But $AC = OA - OC$, which taken from OG, there will remain $OG + OC - OA = OI$; that is, if the Logarithm of the Multiplier and Multiplicand be added together, and from the Sum be taken the Logarithm of Unity, (which is always express'd by 10 or 100 with Cyphers) the Logarithm of the Product will be had. For Example, let the Decimal Fraction 0,00734 be to be multiplied by the Fraction 0,000876. Set down 100 for the Index of the Logarithm of Unity, and then the Logarithms of the Fractions will be as in the Margin; which being added together, and the Logarithm of Unity being taken away

97,8656961
96,9425041
94,8082002

from the Sum, the Remainder is the Logarithm of the Product, whose Index 94 shews, that the first Figure of the Product is in the sixth Place from Unity; and so there must be five Cyphers prefix'd, and then the Product will be, 00000642984.

In Division, the Divisor is to Unity, as the Dividend is to the Quotient; and so the Distance between the Divisor and Unity shall be equal to the Distance between the Dividend and the Quotient. And so, if the Fraction IK be to be divided by DC, you must take $IG = CA$, and the Place of the Quotient shall be G. But $CA = OA - OC$, which being added to OI, we have $OA + OI - OC = OG$; that is, if the Logarithm of Unity be added to the Logarithm of the Dividend, and from the Sum be taken the Logarithm of the Divisor, there will remain the Logarithm of the Quotient; so if the Number CD be to

be

be divided by IK, you must take the Distance CS = IA, and then ST will be the Quotient, whose Logarithm is $OA + OC - OI$. Let $CD = 0.347$, $IK = 0.00478$. Then add the Logarithm of Unity to the Logarithm of CD; 19.5403295 that is, put 1 or 10 before the Index 7.6794270 thereof, and from that subtract the Logarithm of the Divisor, and the Remainder will be the Logarithm of the Quotient, whose Index 11 shews, that the Quotient is between the Numbers 10 and 100; and I seek the Number answering the Logarithm, which I find to be 72,542. If the Logarithm of a Vulgar Fraction, for Example, $\frac{7}{8}$, be requir'd, the Logarithm of Unity must be added to the Logarithm of the Numerator 7; or, which is all one, you must put 10 or 100 before the Index thereof, and subduct from it the Logarithm of the Denominator 8; and there will remain the Logarithm of the Vulgar Fraction $\frac{7}{8}$, or the Decimal .875.

If the Powers of any Fraction DC be requir'd, you must assume EC, EG, GI, IL, each equal to AC; and then EF will be the Square, GH the Cube, and IK the Biquadrate of the Number DC; for they are continually proportional from Unity. Besides, $AE = 2AC = 2AO - 2OC$; whence $OE = OA - AE = 2OC - OA$; that is, the Logarithm of the Square is the Double of the Logarithm of the Root, less the Logarithm of Unity. In like manner, since $AG = 3AC = 3OA - 3OC$, we shall have $OG = OA - AG = 3OC - 2OA =$ the Logarithm of the Cube = triple the Logarithm of the Root, — the Double of the Logarithm of Unity. For the same Reason, because $AI = 4AC = 4OA - 4OC$, we have $OI = 4OC - 3OA$, which is the Logarithm of the Biquadrate. And, universally, if the Power of a Fraction be n , and the Logarithm L, then shall the Logarithm of the Power $n = nL - nOA + OA$; that is, if the Logarithm of a Fraction be multiplied by n , and from the Product be taken the Logarithm of Unity, multiplied by $n-1$, the Logarithm of the Power n of that Fraction will be had.

Of LOGARITHMS.

For Example, if it is required to find the 6th Power of the Fraction $\frac{1}{2} = ,05$ the Logarithm of this Fraction is 8.6989700, which, being multiplied by 6, gives the Number 52.1938200; and if from 52 the Number 50, which is the Index of the Logarithm of Unity drawn into 5, be taken away, the Remainder will be the Logarithm of the 6th Power, viz. 2.1938200, to which the Number ,000000015625 answers. For the Index 2 shews, that 7 Cyphers must be put before the first Figure.

If the 8th Power of the Fraction ,05 be required, by multiplying the Logarithm by 8, there will be produced 69.5917600; and since 70, which is seven times the Index of the Logarithm of Unity, cannot be taken from 69, unless we run into negative Numbers, the Index of the Logarithm of Unity must be supposed 100, and then the Index of the Logarithm of the Fraction will be 98. Now this Logarithm, drawn into 8, gives 789.5917600; and if 700, which is 7 times the Index of the Logarithm of Unity, be taken from 789, there will remain 89.5917600, the Logarithm of the 8th Power of the Fraction $\frac{1}{2}$, whose correspondent Number is ,00000000039062: For since the Index is 89, and the Difference thereof from 100 is 11; the first significative Figure of the Fraction shall be in the 11th Place from Unity; and so there must be 10 Cyphers placed before it.

If the Roots of the Powers of Fractions be desir'd, for Example, the Square Root of the Fraction EF, because the Root is a mean Proportional between the Fraction and Unity, you must bisect AE in C, and then CD will be the Square Root of the Fraction EF.

But $AC = \frac{1}{2} AE = \frac{OA - OE}{2}$; and so the Loga-

rithm of the Root $= OA - AC = \frac{OA + OE}{2}$. And

if the Cube Root of the Fraction GH be sought, this shall be the first of two mean Proportionals between Unity and GH; and so, if AG be divided into three equal Parts, the first of which is AC, then CD shall be the Root sought: And because $AC = \frac{1}{3} AG = \frac{OA - OG}{3}$, if this be taken from OA, there will

remain

remain $\frac{2OA + OG}{3} = OC = \text{Logarithm of the Cube}$

Root of the Fraction GH. So, likewise, the biquadrate Root of the Fraction IK will be had, by dividing AI into four equal Parts; for the Root is the first of three mean Proportionals between Unity and the Fraction; and, consequently, if $AC = \frac{1}{4} AI$, then will CD be the biquadrate Root of the Fraction IK. But $\frac{1}{4} AI = \frac{OA - OI}{4}$; and so $OC =$

$$OA - AC = \frac{3OA + OI}{4}.$$

And universally, if the Root of any Power n of the Fraction LM be requir'd, the Logarithm of the Root thereof will be $\frac{nOA - OA + OL}{n}$; that is, if the

Number $n-1$ be prefix'd to the Index of the Logarithm, and the Logarithm thus augmented be divided by n , the Quotient will give the Logarithm of the Root sought. So if the Cube Root of the Fraction $\frac{1}{2}$ or .5 be sought, you must place $2 = n-1$ (since the Cube Root is required) before the Logarithm thereof, and there will be had 29,6989700, a third Part of which is 9,8996566, which is equal to the Logarithm of the Cube Root of the Fraction $\frac{1}{2}$; and the Number ,7937, answering to this Logarithm, is the Root sought,

CHAP. IV.

Of the Rule of Proportion by Logarithms.

THE Rule of Proportion shews how, by having three Numbers given, a fourth Proportional to them may be found; viz. if the second and third Terms be multiplied by one another, and the Product divided by the first Term, then will the Quotient be the fourth proportional Term sought. But this fourth Term is much easier found by Logarithms; for if the Logarithm of the first Term be taken from the Sum of the Logarithms of the second and third Term, the Num-

ber remaining will be the Logarithm of the fourth sought.

Or this may be done something easier yet, if instead of the Logarithm of the first Term be taken its Complement Arithmetical, or the Difference of that Logarithm, and the Number 10.0000000, which is done by setting down the Difference between each Figure of the Logarithm, and the Figure 9; for then, if that Arithmetical Complement be added to the Sum of the other two Logarithms, and if Unity, which is the first Figure to the Left Hand, be taken from the Sum, the Remainder will be the Logarithm of the fourth Term sought; and so, by this Way, Logarithms of the fourth Term are found by only one Addition of three Numbers. The Reason of this will be manifest from hence: Let there be three Numbers A, B, C, from which the first is to be taken from the Sum of the second and third. Now this may not only be done by the common Way, but, likewise, if there be any other third Number E taken, and from this there be taken A, there will remain $E - A$; and if the Numbers B, C, and $E - A$ be all added together, and from their Sum be taken E, there will remain $B + C - A$. So, if the Number 15 be to be taken from 23, take the Complement of the Number 15 to 100, which is 85, and add this Number to 23, and the Sum will be 108, from which 100 being taken, there remains the Number 8.

Hence follow some Trigonometrical Examples of the Rule of Proportion solved by Logarithms.

Let ABC be a right-angled Triangle, wherein are given, the Angle A 36 Degrees 46', the Angle B 98 Degrees 32', and the Side BC 3478, the Side AC is required. Say (*by Case 1. of plain Trig.*) as the Sine of the Angle A is

to the Sine of the	Arith. Comp. S, A.	0.2228938
Angle B, so is BC	Log. Sin. B.	9.9951656
to AC. And be-	Log. BC.	3.5413296
cause the Loga-	Log. AC.	<u>×3.7593890</u>
rithm Sine of the		

Angle A is the first Term of the Analogy, I substitute its Complement Arithmetical for the same, and add the Logarithm of BC, the Logarithm of S, B, and the said Complement, all three together, and reject Unity,

Unity, which is in the first Place to the Left Hand ; and then the Logarithm of the Side AC will be given, and the Number answering thereto is 5706,306 equal to the Side sought AC.

Let there be a spherical Triangle ABC, in which are given all the Sides, viz. BC = 30 Degrees, AB = 24 Degrees 4', and AC = 42 Degrees 8', the Angle B is required. Let BA be produced to M, so that BM=BC; then will AM, the Difference of the Sides BC, BA, be equal to 5 Degrees 56'. Now (by Case II. in oblique-angled spherical Triangles) say, As the Rectangle under the Sines of the Legs is to the Square of Radius, so is the Rectangle under the Sines of the Arcs $\frac{AC+AM}{2}$, $\frac{AC-AM}{2}$, to the Square of the Sine of one half the Angle B.

But $\frac{AC+AM}{2} = 24$ Degrees 2', and $\frac{AC-AM}{2} = 18$ Degrees 6'; and because the first Term of the Analogy is the Rectangle under the Sines of AB, BC, and second Term is the Square of Radius, the Sum of the Logarithm Sine of AB, BC, must be taken from double the Logarithm of Radius, and what remains must be added to the Sum of the Logarithm S, of $\frac{AC+AM}{2}$, and $\frac{AC-AM}{2}$, which is the same as if the Logarithm Sines of each of the Arcs AB, BC, were sub-

tracted from	Log. S, BC Comp. Arith.	0.3010299
the Log. of	Log. S, AB Comp. Arith.	0.3898364
Radius; or if	Log. S, $\frac{AC+AM}{2}$	9.6098803
the Comple-	Log. S, $\frac{AC-AM}{2}$	9.4923083
ments Arith-	2 Log. S, Angle B.	19.7930549
metical of		
these Sines		
be taken, and		

the Complements and the said Sines be all added together, then shall the Sum be the Logarithm of the Square of the Sine of half the Angle B. And so the Half of the Logarithm 9.8965274 is the Logarithm Sine of half the Angle B=51 Degrees 59' 56", and the Double of this Angle shall be 103 Degrees 59' 52" = B, which was sought.

CHAP. V.

Of the continual Increments of proportional Quantities, and how to find by Logarithms, any Term in a Series of Proportionals, either increasing or decreasing. Fig. 3.

IF any-where in the Axis of the Logarithmetical Curve, there be taken any Number of equal Parts SV, VY, YQ, &c. and at the Points S, V, Y, Q, &c. be raised the Perpendiculars ST, VX, YZ, QΠ, &c. then, from the Nature of the Curve, shall all these Perpendiculars be continually proportional; and therefore, also, the continual Increments Xx, Zz Ππ; shall be proportional to their Wholes: For since $ST : VH :: VX : YZ :: YZ : QΠ$, it shall be (by Division of Proportion) $ST : Xx :: VX : Zz :: YZ : Ππ$; and (by Composition of Proportion) $VX : Xx :: YZ : Zz :: QΠ : Ππ$. Hence, if Xx be any Part of any right Line ST, then will Zz be the same Part of the right Line VX, and also Ππ the same Part of the right Line YZ. For Example; if Xx be the $\frac{1}{10}$ Part of ST, then will $Zz = \frac{1}{10} VX$, and $Ππ = \frac{1}{10} YZ$; or, which comes to the same, we shall have $VX = ST + \frac{1}{10} ST$, $YZ = VX + \frac{1}{10} VX$. Also, $QΠ = YZ + \frac{1}{10} YZ$.

Now make, as ST is to VX, so is Unity AB to NR; then shall $AN = SV$; and so each of the right Lines SV, VY, YQ, &c. shall be equal to the Logarithm of NR; and AV, the Logarithm of the Term VX, shall be equal to $AS + AN =$ Logarithm of $ST +$ Logarithm of NR. Also AY, the Logarithm of the Term YZ, shall be equal to $AS + 2AN =$ Logarithm ST + 2 Logarithm NR; and AQ, the Logarithm of the Term QΠ, shall be equal to $AS + 3AN =$ Logarithm ST + 3 Logarithm NR. And universally, if the Logarithm of the Number NR be multiplied by a Number expressing the Distance of any Term from the first, and the Product be added

to

to the Logarithm of the first Term, then will the Logarithm of that Term be had : But if a Series of Proportionals be decreasing, that is, if the Terms diminish in a continual Ratio, and $Q\Pi$ be the first Term ; then the Logarithm of any other will be had, in multiplying the Logarithm of the Number NR , by a Number that expresses the Distance of its Term from the first, and subtracting the Product from the Logarithm of the first. And if the said Product be greater than the Logarithm of the first Term, then the Logarithms must begin from a Unit in some Place of Decimal Fractions, as from OP , and then the Logarithm of the Number $Q\Pi$ will be OQ .

Now, let LM represent any Money, or Sum of Money, put out to Interest, so that the Interest thereof be accounted but at the End of every Year, and let Kk be the Gain or Interest thereof at the End of the first Year ; then will IK be the Sum of the Interest and Principal. And again, IK becoming the Principal at the End of the first Year, Hh , which is proportional to IK , or in a constant Ratio, will be the Gain at the End of the second Year ; and so HG , at the End of the second Year, will become the Principal ; and at the End of the third Year Ff , proportional to GH , will be the Gain. Now, let us suppose the Principal be augmented every Year $\frac{1}{20}$ Part thereof, so that $IK = LM + \frac{1}{20} LM$, $GA = IK + \frac{1}{20} IK$, $EF = GH + \frac{1}{20} GH$, and so on. And accordingly the Terms LM , IK , GH , EF , &c. continual Proportionals, it is required to find the Amount of the Money at the End of any Number of Years.

Let LM be a Farthing. Because LM is to IK as 1 to $1 + \frac{1}{20}$, or as 1 to 1.05, as AB is to NR , then will $NR = 1.05$, whose Logarithm AN is 0.0211893, or more accurately 0.0211892991, it is required to find the Amount of a Farthing put out at compound Interest, at the End of 600 Years. Multiply AN by 600, and the Product will be 12.7135794, and to this Product add the Logarithm of the Fraction $\frac{1}{960}$, viz. 97.0177288, (for a Farthing is $\frac{1}{960}$ Part of a Pound) and the Sum 109.7313082 shall be the Logarithm of the Number sought ; and since the Index 109 exceeds the Index of Unity by 9, there shall be nine Places of Figures above Unity in the correspondent Number ;

ber; and that Number, being sought in the Tables, will be found greater than 5386500000, and less than 5386600000. And therefore a Farthing put out at Interest upon Interest, at 5 *per Cent. per Annum*, at the End of 600 Years will amount to above 5386500000 Pounds; which Sum could hardly be made up by all the Gold and Silver that has been dug out of the Bowels of the Earth from the Beginning of the World to this Time.

Let $Q\Pi$ expound any Sum of Money due to some Person at the End of a full Year. Now it is certain, that if the Debtor should pay down present the whole Sum of Money, he would lose the yearly Usury or Interest that his Money would gain him; and so a lesser Sum, being put out to Interest, will, at the End of one Year, together with the Interest thereof, be equal to the Sum of Money $Q\Pi$. Now this present Sum of Money, which, together with the rest thereof, is equal to the Sum of Money $Q\Pi$, is called the present Worth of the Money $Q\Pi$. Let AN be the Logarithm of the Ratio which the Principal has to the Sum of the Principal and Interest, that is, if the Principal be twenty times the yearly Interest, let AN be the Logarithm of the Number $1 + \frac{1}{20}$ or 1.05, and take QY equal to AN ; then will AY be the Logarithm of the present Worth of the Money $Q\Pi$. For it is manifest, that the Money YZ put out to Interest, will, at the End of one Year, amount to the Money $Q\Pi$; and so, to have the Logarithm of the present Worth thereof, or YZ , the Logarithm AN must be taken from the Logarithm AQ , and there will remain the Logarithm AY of the present Worth, or YZ . But if the Sum $Q\Pi$ be not due till the End of two Years, then the Logarithm $2AN$ must be subtracted from the Logarithm AQ , and there will remain AV , the Logarithm of the present Worth, or of the Sum that must be paid down present for the Money $Q\Pi$ due at the End of two Years. For it is manifest, that the Money VX being put out to Interest, will, at the End of two Years, amount to the Sum of Money $Q\Pi$. By the same Reason, if the Sum $Q\Pi$ be not due until the End of three Years, the Logarithm $3AN$ must be subtracted from the Logarithm of $Q\Pi$, and the Remainder AS shall be the Logarithm of the Number ST , or

ST shall be the present Worth of the Sum $Q\Pi$ due at the three Years End. And universally, if the Logarithm AN be multiplied by the Number of Years, at the End of which the Sum $Q\Pi$ is due, and the Number produced be taken from the Logarithm AQ, then will the Logarithm of the present Worth of the Sum $Q\Pi$ be had. And from hence it is manifest, if 5386500000 Pounds be due to some Society at the End of 600 Years, then would the present Worth of that vast Sum of Money be scarcely a Farthing.

If the proportional right Lines HG, EF, AB, CD, *Fig. 4.* are Ordinates to the Axis of the Logarithmical Curve, and if their Ends FH, DB, be join'd by right Lines, which, produced, meet the Axis in the Points P and K, then the right Lines GP, AK, will be always equal. For since $GH : EF :: AB : CD$; it will be, as $GH : Fs :: AB : DR$. But because of the equiangular Triangles PGH, HsF, as also KAB, BRD, we have $PG : Hs :: (GH : Fs :: AB : DR ::) KA : BR$. And since the Consequents Hs, BR, are equal, the Antecedents PG, KA, shall be also equal. W. W. D.

If the right Lines CD, EF, equally accede to AB, GH, so that the Point D at last may coincide with B, and the Point F with H, then the right Lines DBK, FHP, which did cut the Curve before, will be changed into the Tangents BT, HV. And the right Lines AT, GV, will be always equal to each other; that is, the Portion of the Axis AT, or GV, intercepted between the Ordinate and the Tangent, which is called the Subtangent, will every-where be a constant and given Length. And this is one of the chief Properties of the Logarithmical Curve; for the different Species or Forms of those Curves are determined by the Subtangents.

The Logarithms, or the Distances from Unity of the same Number, in two Logarithmical Curves of different Species, will be proportional to the Subtangents of their Curves. For let HBD, SNY, *Fig. 4, 5.* be Curves, whose Subtangents are AT, MX, and let $AB = MN = \text{Unity}$; also, $DC = QY$; then shall AC, the Logarithm of the Number CD, in the Logarithmical Curve HD, be to MQ, the Logarithm of the Number QY, (or of the said CD) in the Curve SY, as the Subtangent AT is to the Subtangent MX.

MX. For let there be supposed an infinite Number of mean proportional Terms between AB, CD, or NM, QY, in the Ratio of AB to ab , or MN to mn ; and since $AB = MN$, then will $ab = mn$, as also $bc = no$. And because the Number of proportional Terms in each Figure are equal, they do divide the Lines AC, MQ, into equal Numbers of Parts, the first of which Aa, Mm, and so the said Parts shall be proportional to their Wholes; that is, it will be as $Aa : Mm :: AC : MQ$. And because the Triangles TAB, Bcb, are similar, (for the Part of the Curve Bb nearly coincides with the Portion of the Tangent) as also the Triangles XMN, Non, we have Aa, or Bc : $bc :: TA : AB$.

Also, as no , or $bc : No :: MN$, or $AB : MX$.

Where (by Equality of Proportion) it will be, $Bc : No :: TA : MX :: Aa : Mm :: AC : MQ$; which was to be demonstrated. If AT be called a , since $AB : AT :: bc : Bc$, then will $Bc = \frac{a \times bc}{AB}$.

Hence, if the Logarithm of a Number extremely near Unity, or but a small matter exceeding it, be given, then will the Subtangent of the Logarithmical Curve be had. For the Excess bc is to the Logarithm Bc, as Unity AB is to the Subtangent AT. Or even if there are any two Numbers nearly equal, their Difference shall be to the Difference of their Logarithms, as one of the Numbers is to the Subtangent. For Example, if the Increment bc be ,00000 00000 00001 02255 31945 60259, and Bc or Aa the Logarithm of the Number ab be ,00000 00000 00000 44408 92098 52062. Now if a fourth Proportional be found to the said two Numbers and Unity, viz. 434294481903251, this Number will give the Length of the Subtangent AT, which is the Subtangent of the Curve expressing Briggs's Logarithms.

If a Sum of Money be put out to Interest on this Condition, that a proportional Part of the yearly Rate of Interest thereof be accounted every Moment of Time, viz. so that at the End of the first Moment of Time, or indefinitely small Particle of a Year, the Interest gotten thereby be proportional to that Time; which being added to the Principal, again begets Interest at the End of the second Moment of Time,

and

and then the Principal and this Interest become a Principal, and so on; it is requir'd to find the Amount of that Sum at the Year's End. Let a be nearly the Interest of Unity, or of one Pound. Then, if one whole Year, or 1, gives the Interest a , the indefinitely small Particle of a Year Mm will give the Interest $Mm \times a$, proportional to Mm ; and, accordingly, if Unity be expounded by MN , the first Increment thereof shall be $no = Mm \times a$. This being granted, let a Logarithmical Curve be suppos'd to be describ'd through the Points Nn , whose Axis is OMQ . Then, in this Curve, if the Portion of the Axis MQ expresses the Time, the Ordinate Qy will represent the Money proportionally increasing every Moment, to that Time. For if there be taken ml , &c. $= Mm$, the Ordinates lp , &c. shall be in a Series of continual Proportionals in the Ratio of MN to mn ; that is, they increase in the same Ratio as the Money doth.

Again, let the right Line NX touch the Logarithmical Curve in N , and the Subtangent thereof MX shall be constant and invariable, and the small Triangle Non shall be similar to the Triangle XMN . But it has been prov'd, that the Increment $no = Mm \times a = No \times a$; and so $no : No :: No \times a : No :: a : 1$. But as no is to No , so shall NM be to MX . Wherefore it shall be, as a is to 1, so is NM , or 1, to $MX = \frac{1}{a} = \text{Subtangent}$.

Now if the nearly Rate of Interest be $\frac{1}{20}$ Part of the Principal, or if $a = \frac{1}{20} = .05$, then will $MX = \frac{1}{a} = 20$.

Because of different Forms of Logarithms, the Logarithms of the same Number are proportional to the Subtangents of their Curves: If MQ expresses the Time of a whole Year, or Unity, then shall QY be the Amount of the Money at the Year's End. And to find QY , say, As MX , or $\frac{1}{a}$, is to 0.4342944, (which Number expounds the Subtangent of the Logarithmical Curve expressing *Briggs's* Logarithms) so is one Year, or Unity, to a *Briggian* Logarithm, answering to the Number QY . This Logarithm will be found 0.0217147, and the Number answering to the same is $1.05127 = QY$, whose Increment above Unity,

Unity, or the Principal, exceeds the yearly Interest .05 but a small Matter. And so if the yearly Interest of 100 Pounds be 5 Pounds, the proportional yearly Interest, which is added to the Principal 100 at the End of each Particle of the Year, will amount only at the Year's End to 5 Pounds 2 Shillings and 6½ Pence.

And if such a Rate of Interest be required, that every Moment a Part of it continually proportional to the increasing Principal be added to the Principal, so that at the Year's End an Increment be produced that shall be any given Part of the Principal; for Example, the $\frac{1}{20}$ Part; say, As the Logarithm of the Number 1.05 is to 1; that is, as 0.0211893 is to 1; so is the Subtangent 0.432944 to $\frac{1}{a} = 20.49$, and

then will $a = \frac{1}{20.49} = .0488$. For if such a Part of the Rate of Interest .0488 be supposed, as answers to a Moment, that is, having the same Ratio to .0488 as a Moment has to a Year, and it be made as Unity is to that Part of the Rate of Interest, so is the Principal to the momentaneous Increment thereof; then will the Money, continually increasing in that manner, be augmented at the Year's End the $\frac{1}{20}$ Part thereof.

C H A P. VI.

Of the Method by which Mr. Briggs computed his Logarithms, and the Demonstration thereof.

ALthough Mr. Briggs has no-where describ'd the Logarithmical Curve, yet it is very certain, that from the Use and Contemplation thereof, the Manner and Reason of his Calculations will appear. In any Logarithmical Curve HBD, let there be three Ordinates AB, ab , qs , nearly equal to one another; that is, let their Differences have a very small Ratio to the said Ordinates; and then the Differences of their Logarithms will be proportional to the Differences of the Ordinates. For since the Ordinates are nearly equal to one another, they will be very nigh

to each other; and so the Part of the Curve Bs , intercepted by them, will almost coincide with a straight Line; for it is certain, that the Ordinates may be so near to each other, that the Difference between the Part of the Curve, and the right Line subtending it, may have to that Subtense a Ratio less than any given Ratio. Therefore the Triangles Bcb , Brs , may be taken for right-lined, and will be equiangular. Wherefore, as $sr : bc :: Br : Bc :: Aq : Aa$; that is, the Excesses of the Ordinates or Lines above the least, shall be proportional to the Differences of their Logarithms. And from hence appears the Reason of the Correction of Numbers and Logarithms by Differences and proportional Parts. But if AB be Unity, the Logarithms of Numbers shall be proportional to the Differences of the Numbers.

If a mean Proportional be found between 1 and 10, or, which is the same thing, if the Square Root of 10 be extracted, this Root or Number will be in the middle Place between Unity and the Number 10, and the Logarithm thereof shall be $\frac{1}{2}$ of the Logarithm of 10, and so will be given. If, again, between the Number before found, and Unity, there be found a mean Proportional, which may be done in extracting the Square Root of the said Number, this Number, or Root, will be twice nearer to Unity than the former, and its Logarithm will be one Half of the Logarithm of that, or one Fourth of the Logarithm of 10. And if in this manner the Square Root be continually extracted, and the Logarithms bisected, you will at last get a Number, whose Distance from Unity shall be less than the $\frac{1}{10000000000}$ Part of the Logarithm of 10. And after Mr. Briggs had made 54 Extractions of the Square Root, he found the Number 1.00000 00000 00000 12781 91493 20032 3442; and its Logarithm was 0. 00000 00000 00000, 05551 11512 31257 82702. Suppose this Logarithm to be equal to Aq or Br, and let qs be the Number found by extracting the Square Root; then will the Excess of this Number above Unity, viz. rs = ,00000 00000 00000, 12781 91493 20032 34.

Now, by means of these Numbers, the Logarithms of all other Numbers may be found in the following manner: Between the given Number (whose Logarithm

rithm

rithm is to be found) and Unity, find so many mean Proportionals, (as above) till at last a Number be gotten so little exceeding Unity, that there be 15 Cyphers next after it, and a like Number of significative Figures after those. Let this Number be ab , and let the significative Figures, with the Cyphers prefixed before them, denote the Difference bc . Then say, As the Difference rs is to the Difference bc , so is Br , a given Logarithm, to Bc , or Aa , the Logarithm of the Number ab ; which therefore is given. And if this Logarithm be continually doubled, the same Number of Times as there were Extractions of the Square Root, you will at last have the Logarithm of the Number sought. Also, by this way may the Subtangent of the Logarithmical Curve be found, *viz.* in saying, $Asrs : Br :: AB$, or Unity : AT , the Subtangent, which therefore will be found to be 0.4342944819-03251; by which may be found the Logarithms of other Numbers; to wit, if any Number NM be given afterwards, as also its Logarithm, and the Logarithm of another Number sufficiently near to NM , be sought, say, As NM is to the Subtangent XM , so is no , the Distance of the Numbers, to No , the Distance of the Logarithms. Now, if NM be Unity = AB , the Logarithms will be had by multiplying the small Differences bc by the constant Subtangent AT .

By this way may be found the Logarithms of 2, 3, and 7; and by these the Logarithms of 4, 8, 16, 52, 64, &c. 9, 27, 81, 243, &c. as also 7, 49, 343, &c. And if from the Logarithm of 10 be taken the Logarithm of 2, there will remain the Logarithm of 5; so there will be given the Logarithms of 25, 125, 625, &c.

The Logarithms of Numbers compounded of the aforesaid Numbers, *viz.* 6, 12, 14, 15, 18, 20, 21, 24, 28, &c. are easily had by adding together the Logarithms of the component Numbers.

But since it was very tedious and laborious to find the Logarithms of the prime Numbers, and not easy to compute Logarithms by Interpolation, by first, second, and third, &c. Differences; therefore the great Men, Sir *Isaac Newton*, *Mercator*, *Gregory*, *Wallis*, and, lastly, *Dr. Halley*, have publish'd infinite converging Series, by which the Logarithms of Numbers

Numbers to any Number of Places may be had more expediently and truer: Concerning which Series Dr. *Halley* has written a learned Tract, in the *Philosophical Transactions*, wherein he has demonstrated those Series after a new Way, and shews how to compute the Logarithms by them. But I think it may be more proper here to add a new Series, by means of which may be found easily and expeditiously the Logarithms of large Numbers.

Let z be an odd Number, whose Logarithm is sought; then shall the Numbers $z - 1$ and $z + 1$ be even, and accordingly their Logarithms, and the Difference of the Logarithms, will be had, which let be called y : Therefore, also, the Logarithm of a Number, which is a Geometrical Mean between $z - 1$ and $z + 1$, will be given, *viz.* equal to the half Sum of the Logarithms. Now the Series $y \times \frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5} + \frac{181}{15120z^7} + \frac{13}{25200z^9}, \text{ \&c.}$ shall be equal to the Logarithm of the Ratio, which the Geometrical Mean between the Numbers $z - 1$ and $z + 1$, has to the Arithmetical Mean, *viz.* to the Number z .

If the Number exceeds 1000, the first Term of the Series $\frac{y}{4z}$ is sufficient for producing the Logarithm to 13 or 14 Places of Figures, and the second Term will give the Logarithm to 20 Places of Figures. But, if z be greater than 10000, the first Term will exhibit the Logarithm to 18 Places of Figures; and so this Series is of great Use in filling up the Logarithms of the Chiliads omitted by *Briggs*. For Example; It is required to find the Logarithm of 20001. The Logarithm of 20000 is the same as the Logarithm of 2 with the Index 4 prefixed to it; and the Difference of the Logarithms of 20000 and 20002 is the same as the Difference of the Logarithms of the Numbers 10000 and 10001, *viz.* 0.00004 34272 7687. And if this Difference be divided by $4z$, or 80004, the Quotient $\frac{y}{4z}$ shall

A a

shall be - - - - - 0.00000 00005 42813
 And if the Logarithm of the 4.30105 17093 02416
 Geometrical Mean be added 4.30105 17098 45231
 to the Quotient, the Sum

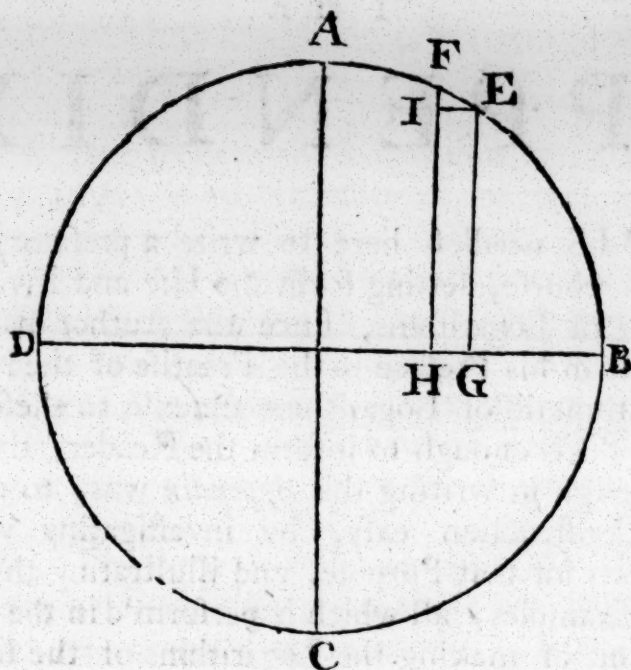
will be the Logarithm of 20001. Wherefore it is manifest, that to have the Logarithm to 14 Places of Figures, there is no Necessity of continuing out the Quotient beyond six Places of Figures. But if you have a mind to have the Logarithm to 10 Places of Figures only, as they are in *Vlaq's* Table, the two first Figures of the Quotient are enough. And if the Logarithms of the Numbers above 20000 are to be found by this Way, the Labour of doing them will mostly consist in setting down the Numbers.

Note, This Series is easily deduced from that found out by *Dr. Halley*; and those who have a Mind to be inform'd more in this Matter, let them consult his above-nam'd Treatise.



T H E
A P P E N D I X.

'TIS needless here to write a prefatory Discourse; setting forth the Use and Invention of Logarithms; since the Author has supplied that in his Preface to the Treatise of the Nature and Arithmetic of Logarithms annex'd to these Elements: 'Tis enough to inform the Reader, that my chief Design in writing this *Appendix* was, to render their Construction easy, by investigating various Theorems for that Purpose, and illustrating them by proper Examples; all which is perform'd in the actual Operation of making the Logarithms of the first 10 Numbers; and of the prime Number 101, which is more than sufficient to inform the meanest Capacity how to examine or construct the whole Table. I have also shewn how, from the Logarithm given, to find its corresponding Number; and the Investigation of the Series omitted by the Author in Page 355. for expeditiously finding the Logarithms of large Numbers. As to those Series exhibited by him in his Trigonometrical Treatise, Page 287. for making the Sines and Cosines; I must declare, that I have exceeded my first Intentions, which were to give their Investigation only; but considering, that as they depended upon the *Newtonian* Series, without the Investigation of which our Author's Series could never be thoroughly understood, I thought it would therefore prove acceptable, if I shew'd their Investigations too, from which those of our Author's easily flow. In order to which, and to keep the Reader no longer in Suspense; let r be put for the Radius of the Circle $ABCD$; a for the Arch BE , whose Length is to be investigated; S equal to the Sine; and v the versed Sine of that Arch; then is $FE = a$, $IF = s$, and $IE = GH = v$.



Whence $\ddot{a} \dot{a} = \ddot{s} \dot{s} + \ddot{v} \dot{v}$; but from the Equation of the Curve, viz. $2rv - vv = ss$ we have $v = r - \sqrt{rr - ss}$.

Therefore $\dot{v} = \frac{\ddot{s} \dot{s}}{rr - ss} \frac{1}{2}$ and $\dot{v} \dot{v} = \frac{2 \ddot{s} \dot{s}}{rr - ss}$; which

being substituted for $\dot{v} \dot{v}$ in the first Equation, we have $\dot{a} = \frac{s^2 \ddot{s}}{rr - ss} + \ddot{s} \dot{s} = \frac{2 \ddot{s} \dot{s}}{rr - ss} = \frac{r \dot{s}}{rr - ss^{\frac{1}{2}}} = r \dot{s} \times \frac{1}{rr - ss} - \frac{1}{2}$.

But $rr - ss^{\frac{1}{2}}$ by Sir Isaac Newton's Binomical Theorem is $r^{-1} + \frac{1}{2} s^2 r^{-3} + \frac{3}{8} s^4 r^{-5} + \frac{5}{48} s^6 r^{-7}$, &c.

Wherefore $r \dot{s} \times \frac{1}{rr - ss^{\frac{1}{2}}} = s + \frac{s^2 s}{2r^2} + \frac{3s^4 s}{8r^4} + \frac{15s^6 s}{48r^6}$, &c. and the Fluent thereof, viz. $s + \frac{s^3}{6r^2} + \frac{3s^5}{40r^4} + \frac{15s^7}{336r^6}$, &c. = $s + \frac{s^3}{2 \cdot 3 r^2} + \frac{3s^5}{2 \cdot 4 \cdot 5 r^4} + \frac{3 \cdot 5 s^7}{2 \cdot 4 \cdot 6 \cdot 7 r^6}$, &c. or $s + \frac{s}{2 \cdot 3 r^2} + \frac{3 \cdot 3 s^5}{2 \cdot 3 \cdot 4 \cdot 5 r^4} + \frac{3 \cdot 3 \cdot 5 \cdot 5 s^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 r^6}$, &c. is equal to the Arch of a Circle

whose



whose Radius is r , and Sine s . But if r be put equal to Unity, then $s + \frac{1}{2.3} s^3 + \frac{3.3}{2.3.4.5} s^5 + \frac{3.3.5.5}{2.3.4.5.6.7} s^7$, &c. will express the Length of the Arch a .

EXAMPLE.

Let it be required to find the Length of the Arch of 30 Degrees, to 6 Places of Decimals, the Radius being Unity.

Here $s = \frac{1}{2}$, and $ss = \frac{1}{4}$; whence the Operation may be as follows:

S	=	5000000	S	=	5000000
S ³	=	1250000	$\frac{1}{6} S^3$	=	208333
S ⁵	=	312500	$\frac{3}{40} S^5$	=	23437
S ⁷	=	78125	$\frac{5}{112} S^7$	=	3487
S ⁹	=	19531	$\frac{35}{1152} S^9$	=	593
S ¹¹	=	4882	$\frac{63}{2112} S^{11}$	=	109
S ¹³	=	1220	$\frac{231}{13312} S^{13}$	=	21
S ¹⁵	=	305	$\frac{1343}{13312} S^{15}$	=	4
					523598.4

Hence the Length of the Arch of 30 Degrees is 523598+. Now if this Arch be multiplied by 6, we shall have the Length of the Arch of the Semicircle in such Parts as the Radius is 1, or of the whole Circumference in such Parts as the Diameter is 1, viz, 3,14159+.

But there is no Series so easy to be retained in the Memory, and so readily put in Practice, for obtaining the Ratio of the Diameter of the Circle to its Circumference, as that which is derived from the Tangent first given. For if t be put equal to the Tangent of any Arch, then $a = t \frac{1}{3} t^3 + \frac{1}{5} t^5 - \frac{1}{7} t^7 + \frac{1}{9} t^9$, &c.

Now the Radius being Unity, the Sine of 30 Degrees = $\frac{1}{2}$, and consequently the Cofine = $\sqrt{\frac{3}{4}}$; and because the Cofine is to the right Sine, as the Radius to the Tangent; it will be, $\sqrt{\frac{3}{4}} : \sqrt{\frac{1}{4}} :: 1 : \sqrt{\frac{1}{2}}$, the Tangent of 30° 00' = t , whence $tt = \frac{1}{2}$: Wherefore, if the Root of $\frac{1}{2}$ be divided continually by 3, and the several Quotients by all the odd Powers successively,

A a 3

viz.

viz. the first by three, the second by 5, &c. the Sum of the affirmative Quotients made less by all the negative ones, will be the Arch of 30 Degrees.

And because the Arch of 30 Degrees is $\frac{1}{6}$ part of the Semi-circumference, if instead of $\sqrt{\frac{1}{3}}$ be taken $6\sqrt{\frac{1}{3}} = \sqrt{12}$, we shall have the Semi-circumference in such Parts as the Radius is Unity; or the whole Circumference, the Diameter being Unity,

The OPERATION stands thus:

$\sqrt{12} = 3,494101$	$+ 3,464101$	
3) 1154700		(— ,384900
5) 384900	(+ 76980	
7) 128300		(— 18328
9) 42766	(+ 4751	
11) 14255		(— 1295
13) 4751	(+ 365	
15) 1583		(— 105
17) 527	(+ 31	
19) 175		(— 0
	$+ 3,546228$	$— ,404637$

Whence $3,546228 - ,404637 = 3,141591$ the same as before. The Impossibility of expressing the exact Proportion of the Diameter of a Circle to its Circumference, by any receiv'd way of Notation, has put the most celebrated Men in all Ages upon approximating the Truth as near as possible; there being a Necessity of a near Quadrature, inasmuch as it is the Basis upon which the most useful Branches of the Mathematics are built. And after the famous *Van Ceulen*, who carried it to 36 Places of Decimals, which he order'd to be engraven on his Tomb-stone, thinking he had set Bounds to farther Improvements; the first that attempted it with Success was the most indefatigable Mr. *Abraham Sharp*, who by a double Computation, *viz.* from the Sine of 6 Degrees one way, and from the Sine and Cosine of 12 Degrees another way, carried it to twice the Number of Places that *Van Ceulen* had done, *viz.* 72.

And since that Time Mr. *Machin*, the present Professor of Astronomy in *Gresham-College*, and Secretary to the Royal Society, by a different Method of Computation, has carried it to 100 Places, almost triple the

Number that *Van Ceulen* had done, which not only confirms *Mr. Sharp's* Quadrature, but shews us, that if the Diameter be 100000, &c. the Circumference will be 3,14159. 26535. 89793. 23846. 26433. 83279. 50288. 41971. 69399. 37510. 58209. 74944. 59230. 78164. 05286. 20899. 86280. 34825. 34211. 70679+ of the same Parts.

Which is a Degree of Exactness far surpassing all Imagination, being by Estimation more than sufficient to calculate the Number of Grains of Sand that may be comprehended within the Sphere of the fix'd Stars.

The late *Mr. Gunn's* Series for determining the Periphery of an Ellipsis (who was my Predecessor in the Mathematical School erected by *Frederic Slare*, M. D. and establish'd by a Decree of the High Court of Chancery for qualifying Boys for the Sea-Service) being new and curious, this Opportunity is taken of making it public.

Let *A* be equal to a Quadrant of the Circle circumscribing the Ellipsis, whose Periphery is requir'd. Then *AX*

$$1 - \frac{1 \cdot 1}{2 \cdot 2} e^2 - \frac{1 \cdot 3 \cdot 1}{2 \cdot 4 \cdot 8} e^4 - \frac{1 \cdot 3 \cdot 5 \cdot 1}{2 \cdot 4 \cdot 6 \cdot 16} e^6 - \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 128} e^8$$

$$\frac{1 \cdot 3 \cdot 7 \cdot 9 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 256} e^{10} - \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 21}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 1024} e^{12}, \text{ \&c.}$$

is the Periphery of a Quadrant of the Ellipsis, where

$$ee = \frac{tt - cc}{tt}, \text{ } t \text{ being the Semi-transverse Diameter,}$$

and *c* the Semi-conjugate.

When this Series came to hand, it was imperfect, inasmuch as there were only the first five Terms without the Law of Continuation: But being desirous of rendering it complete, after some Consideration I found the Law to be as follows: It is plain by Inspection, that the Numerators and Denominators of each Term are compos'd of Numbers that run in arithmetical Progression, except the last in each Term, viz. 2, $\frac{1}{8}$, $\frac{1}{6}$, $\frac{1}{4}$, &c. and those being found by the continual Multiplication of these Fractions, $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{4} \times \frac{3}{8} \times \frac{5}{8} \times \frac{7}{16} \times \frac{9}{16}$, &c. the Law of continuing the whole Series as above, is evident. Whence, by a well-known Method of substituting Capital Letters for each Term respectively, the

following Series is deduced, viz. *AXI* — $\frac{1 \cdot 3}{4 \cdot 4} e^2$

$$B - \frac{3.5}{6.6} e^2 C - \frac{5.7}{8.8} e^2 D - \frac{7.9}{10.10} e^2 E - \frac{9.11}{12.12} e^2 F,$$

ℳc. where the Law of Continuation is evident also, since each Capital Letter is equal to its precedent Term,

$$\text{viz. } B = \frac{1}{4} ee, C = \frac{1.3}{4.4} e^2 B, \text{ \&c. and without doubt}$$

in Practice is preferable to the former Series: But the Investigation of that, on which this last depends, is omitted; purely on account of its being foreign to the present Subject.

But to return; if the Series expressing the Length of the Arch, viz. $s + \frac{1}{6}s^3 + \frac{3}{40}s^5$, ℳc. be revers'd, we shall have the Value of s in the Terms of a , and consequently a direct Method for finding the Sine of any Arch from its Length given. Thus,

$$\text{If } a = s + \frac{1}{6}s^3 + \frac{3}{40}s^5 + \frac{1}{112}s^7, \text{ \&c.}$$

$$\text{Then } s = a - \frac{1}{6}a^3 + \frac{1}{120}a^5 - \frac{1}{140}a^7, \text{ \&c.}$$

$$\text{Or } s = a - \frac{a^3}{2.3} + \frac{a^5}{2.3.4.5} - \frac{a^7}{2.3.4.5.6.7}, \text{ \&c.}$$

$$\text{For put } s = Aa + Ba^3 + Ca^5, \text{ \&c.}$$

$$\text{Then } \frac{1}{6}s = \frac{1}{6}A^3a^3 + \frac{1}{2}A^2Ba^5, Ba^5, \text{ \&c.} \quad \left. \begin{array}{l} \text{Then } \frac{1}{6}s = \frac{1}{6}A^3a^3 + \frac{1}{2}A^2Ba^5, Ba^5, \text{ \&c.} \\ \text{And } \frac{3}{40}s^3 = \frac{3}{40}A^3a^3, a^5, \text{ \&c.} \end{array} \right\} = a.$$

$$\text{And } \frac{3}{40}s^3 = \frac{3}{40}A^3a^3, a^5, \text{ \&c.}$$

And consequently $Aa = a$, and $A = 1$, also $B + \frac{1}{6}A = 0$, and $B = -\frac{1}{6}$, $A^3 = -\frac{1}{6}$, also $C + \frac{1}{2}A^2B + \frac{3}{40}A^3 = 0$, and $C = -\frac{1}{2}A^2B - \frac{3}{40}A^3 = -\frac{1}{2} \times -\frac{1}{6} - \frac{3}{40} = \frac{1}{12} - \frac{3}{40} = \frac{1}{120}$. Wherefore $A = 1$, $B = -\frac{1}{6}$, $C = \frac{1}{120}$, ℳc. and consequently, $s = a - \frac{a^3}{6}$

$+ \frac{a^5}{120}$, ℳc. From which three Terms the Law of

Continuation is easily discover'd, making $s = a - \frac{a^3}{2.3}$

$$+ \frac{a^5}{2.3.4.5} - \frac{a^7}{2.3.4.5.6.7} + \frac{a^9}{2.3.4.5.6.7.8.9}, \text{ \&c.}$$

Whence substituting A for a , and we shall have $A -$

$$\frac{A^3}{1.2.3} + \frac{A^5}{1.2.3.4.5} - \frac{A^7}{1.2.3.4.5.6.7} + \frac{A^9}{1.2.3.4.5.6.7.8.9},$$

ℳc. for the *Newtonian* Series, according to our Author's Form, for finding the Sine of any Arch, its Length being given. Q. E. I.

Again,

Again, because the Square of the Radius made less by the Square of the Sine, is equal to the Square of the Cofine, by the second Proposition of our Author's Elements of Plain Trigonometry ; it follows, that if from the Square of the Radius = 1 be taken the Square of the Sine = $a - \frac{1}{6}a^3 + \frac{1}{120}a^5$, &c. the Square Root of the Remainder will be the Cofine = $1 - \frac{1}{2}a^2 + \frac{1}{24}a^4 - \frac{1}{720}a^6$, &c. thus,

$$S = a - \frac{1}{6}a^3 + \frac{1}{120}a^5, \text{ \&c.}$$

$$S = a - \frac{1}{6}a^3 + \frac{1}{120}a^5, \text{ \&c.}$$

$$a^2 - \frac{1}{6}a^4 + \frac{1}{120}a^6, \text{ \&c.}$$

$$- \frac{1}{6}a^4 + \frac{1}{35}a^6, \text{ \&c.}$$

$$+ \frac{1}{120}a^6, \text{ \&c.}$$

$Ss = a^2 - \frac{1}{3}a^4 + \frac{2}{45}a^6$, &c. which, being taken from the Square of the Radius 1, leaves $1 - aa + \frac{1}{3}a^2 - \frac{2}{45}a^4$, &c. the Square Root of which will be the Cofine.

$$1 - aa + \frac{1}{3}a^2 - \frac{2}{45}a^4, \text{ \&c.} \quad \left(1 - \frac{1}{2}a^2 + \frac{1}{24}a^4 - \frac{1}{720}a^6, \text{ \&c.} \right)$$

$$\begin{array}{r} 1 \\ 2 - aa) \quad -aa + \frac{1}{3}a^2 \\ \quad -aa + \frac{1}{3}a^2 \\ \hline 2 - aa, \text{ \&c.} \quad \frac{1}{12}a^4 - \frac{2}{45}a^6, \text{ \&c.} \\ \quad \frac{1}{12}a^4 - \frac{1}{24}a^6, \text{ \&c.} \\ \hline 2 - aa, \text{ \&c.} \quad -\frac{1}{30}a^6, \text{ \&c.} \\ \quad \frac{1}{30}a^6, \text{ \&c.} \\ \hline 0 \end{array}$$

Wherefore putting A for a, we shall have for the Cofine

$$1 - \frac{A^2}{2} + \frac{A^4}{24} - \frac{A^6}{720}, \text{ \&c.} \quad \text{or} \quad 1 - \frac{A^2}{1.2} + \frac{A^4}{1.2.3.4} - \frac{A^6}{1.2.3.4.5.6} + \frac{A^8}{1.2.3.4.5.6.7.8}, \text{ \&c.}$$

according to the Author's Form. Q. E. I.

But because those Series, as our Author observes, converge very slowly, especially when the Arch is nearly equal to the Radius; he therefore devised (Page 287.) other Series, whose Investigation may be as follows:

Let

The APPENDIX.

Let the Arc, whose Sine is sought, be the Sum or Difference of two Arcs, viz. $A+z$, or $A-z$; and let the Sine of the Arc A be call'd a , and the Cofine b .

Now, if the Arc $DF = DE$, Prop. 5th of the Elements of Trigonometry, be call'd z , then its Sine FO

will, by the *Newtonian* Series, be $= z - \frac{z^3}{1.2.3} +$

$\frac{z^5}{1.2.3.4.5} - \frac{z^7}{1.2.3.4.5.6.7}$, &c. and its Cofine CO

$= 1 - \frac{z^2}{1.2} + \frac{z^4}{1.2.3.4} - \frac{z^6}{1.2.3.4.5.6}$, &c. and because

$CD : DK :: CO : OP$; therefore $OP = a - \frac{az^2}{1.2} + \frac{az^4}{1.2.3.4} - \frac{az^6}{1.2.3.4.5.6}$, &c.

Again, because the Triangles CDK, FOM , are similar, it will be, as $CD : CK :: FO : FM$; whence

$FM = \frac{bz}{1} - \frac{bz^3}{1.2.3} + \frac{bz^5}{1.2.3.4.5} - \frac{bz^7}{1.2.3.4.5.6.7}$, &c.

But $OP + FM = IF$, the Sine of the Arc BF ,

viz. $A+z$; consequently the Sum of those Series,

viz. $a + \frac{bz}{1} - \frac{az^2}{1.2} - \frac{bz^3}{1.2.3} + \frac{az^4}{1.2.3.4} + \frac{bz^5}{1.2.3.4.5}$, &c.

is the Sine of the Arc $A+z$. And because $FM = MG$, therefore their Difference $a - \frac{bz}{1} - \frac{az^2}{1.2} + \frac{bz^3}{1.2.3}$

$+ \frac{az^4}{1.2.3.4} - \frac{bz^5}{1.2.3.4.5}$, &c. is the Sine of the Arc

$A-z$, viz. EL .

And again, because $CD : CK :: CO : CP$;

therefore $CP = b - \frac{bz^2}{1.2} + \frac{bz^4}{1.2.3.4} - \frac{bz^6}{1.2.3.4.5.6} +$

$\frac{bz^8}{1.2.3.4.5.6.7.8}$, &c. And by reason of the similar Tri-

angles CDK, FMO , it will be, as $CD : DK :: FO :$

MO . Whence $MO = az - \frac{az^3}{1.2.3} + \frac{az^5}{1.2.3.4.5} -$

$\frac{az^7}{1.2.3.4.5.6.7}$, &c.

But $CP - MO = CI$, the Cofine of the Arc $A+z$.
Where-

Wherefore the Cosine of the Arc $A + z$ is $b - \frac{az}{1} - \frac{bz^2}{1.2} + \frac{az^3}{1.2.3} - \frac{bz^4}{1.2.3.4} + \frac{az^5}{1.2.3.4.5} - \frac{bz^6}{1.2.3.4.5.6}$

&c.

And because $MO = PL$, therefore $CP + MO = CL$, and consequently the Cosine of the Arc $A - z = b + \frac{az}{1} - \frac{bz^2}{1.2} + \frac{az^3}{1.2.3} - \frac{bz^4}{1.2.3.4} + \frac{az^5}{1.2.3.4.5}$, &c.

Q. E. I.

Now the Arc A is an Arithmetical Mean between the Arcs $A - z$ and $A + z$, and the Difference of their

Sines are $\frac{bz}{1} - \frac{az^2}{1.2} - \frac{bz^3}{1.2.3} + \frac{az^4}{1.2.3.4} + \frac{bz^5}{1.2.3.4.5} - \frac{az^6}{1.2.3.4.5.6}$, &c. $\frac{bz}{1} + \frac{az^2}{1.2} - \frac{bz^3}{1.2.3} - \frac{az^4}{1.2.3.4} + \frac{bz^5}{1.2.3.4.5} + \frac{az^6}{1.2.3.4.5.6}$, &c. Whence the Difference of

the Differences, or second Difference, is $\frac{2az^2}{1.2} - \frac{2az^4}{1.2.3.4} + \frac{2az^6}{1.2.3.4.5.6}$, &c. or $2a \times \frac{z^2}{1.2} - \frac{z^4}{1.2.3.4} + \frac{z^6}{1.2.3.4.5.6}$

&c. Which Series is equal to double the Sine of the mean Arc, drawn into the versed Sine of the Arc z , and converges very soon; so that if z be the Arch of the first Minute of the Quadrant, our Author says the first Term of the Series gives the second Difference to 15 Places of Figures, and the second Term to 25 Places.

Whence the following Rule is deriv'd for finding the Sine of the Arc $A + z$, or $A - z$,

R U L E,

From double the Sine of the mean or middle Arc, subtract the second Difference found by the Theorem, and from the Remainder subtract the Sine of the given Extreme, whether it be the greater or least; and the Remainder will be the Sine of the other Extreme.

E X A M P L E.

Let it be requir'd to find the Sine of $30^\circ 01'$, the Sines of $30^\circ 00'$, and $29^\circ 59'$ being both given.

Here

Here $30^{\circ} 00'$ is the mean Arc, whose Sine is 500000 00000, and the Sine of $29^{\circ} 59'$, the given Extreme, is 4,9974806226, and the Length of the Arc z , viz. one Minute, is ,000290888208; which squar'd and multiplied by the Sine of the mean Arc 50000, &c. according to the Direction of the Theorem, the Product will be the second Difference, equal to ,000000042307; which subtracted from double the Sine of the mean Arc, equal to 1, the Remainder will be ,999999957693; from which subtract the Sine of the given Extreme, (which in this Case is the least) and there will remain 5002518943 for the Sine of $30^{\circ} 01'$, the greater Extreme.

This Method of making the Sines, however it may appear at first Sight, is so far from being tedious or troublesome, that I look upon it to be the most eligible of any other whatsoever; for the Square of z being once determin'd, and the several Multiples of it by the nine Digits made, and set down in a Table orderly, all the Sines may be made by Addition and Subtraction only; as indeed our Author hints they may by the Method demonstrated in the 10th Proposition of the Elements of Trigonometry; but this is evidently preferable to that, tho' a good Method too; and by which all the Sines of the Quadrant, I presume, were wont to be made, at least as far as 30, or 60 Degrees; for after the Sines as far as 60 Degrees are obtain'd, all the others may be had by Addition only; and notwithstanding there are other excellent Theorems, which contribute very much towards finishing and confirming the Truth of the whole Canon; yet this deduced from our Author's Series, I deem the most elegant and fit for Practice; because the Difference of the Differences of the Sines being what is always required to be found, there will be seven Cyphers at least before the significant Figures of the said Difference; which is the Product made by the Square of z , into the Sine of the mean Arc: So that to have the Sine true to ten Places, there will not be occasion to find above four or five Figures in the Product, which, according to the common Method of contracted Multiplication, may be obtained with very few Figures. Thus, for Instance, the Sine of $30^{\circ} 02'$ may be had to ten Places by a wonderful easy Operation, the Sines of $30^{\circ} 01'$ and $30^{\circ} 00'$ being both given.

EXAMPLE.

The Sine of $30^{\circ} 01'$ is	,50025189543
The Square of z inverted	<u>16480000000</u>
	40020
	2001
	300
	<u>5</u>
	<u>42326</u>

Whence the Product is ,000000042326 true to eleven Places at least. Wherefore if, according to the Rule, from double the Sine of the middle Arc = 1,00050379086 we subtract the said Product,

And from the Remainder	1,00050374854
the Sine of $30^{\circ} 00'$ the given Extreme	<u>,50000000000</u>
be subtracted	<u>,50050374854</u>

There will remain ,50050374854 for the Sine of $30^{\circ} 02'$ the other Extreme; than which, nothing of this Nature can be desired more easy.

SCHOLIUM.

Because the Difference of the Differences of the Sines, or second Difference, has always 7 Cyphers before the significant Figures; it follows, that the whole Canon, where the Sines consist but of 6 Places, which is as far as our Tables for common Practice need extend, may be performed chiefly by Addition and Subtraction only, without forming Multiples of the Square of z by the nine Digits; tho' perhaps it may be necessary to use the Method of contracted Multiplication every 5th Minute, to confirm the Truth, lest, in continual doubling and subtracting, an Error should arise in the right-hand Figure: However, as it may be safely used for 5 Minutes together, and sometimes more, it will render the Whole very easy.

Note, The Square of z in this Case, viz. the Arch of 5 Minutes, is ,00000211.

Thus

Thus having investigated the *Newtonian* and our Author's Series, and exemplified the latter, by making the Sines of $30^{\circ} 01'$ and $30^{\circ} 02'$, and withal shewn how, from the Sine of an Arc given, to find the Length of that Arc, and consequently the Circumference of the whole Circle; I shall beg leave, before I treat of the Construction of Logarithms, to shew how, from the known Ratio of the Diameter to the Circumference, or any other Ratio whatsoever, that a Set of integral Numbers may be found, whose Ratios shall be the nearest possible to the Ratio given; for which I hope to be excus'd, and the rather, because I believe this Method of determining them was never before publish'd.

R U L E.

Divide the Consequent by the Antecedent, and the Divisor by the Remainder, and the last Divisor by the last Remainder, and so on till nothing remains.

Then for the Terms of the first Ratio, Unity will always be the Antecedent, and the first Quotient the first Consequent.

For the TERMS of the second RATIO.

Multiply the last $\left\{ \begin{array}{l} \text{Antecedent} \\ \text{Consequent} \end{array} \right\}$ by the 2d Quotient, and to the Product add $\left\{ \begin{array}{l} \text{Nothing;} \\ \text{Unity;} \end{array} \right\}$ and so will the Result be the second $\left\{ \begin{array}{l} \text{Antecedent.} \\ \text{Consequent.} \end{array} \right\}$

For all the following RATIOS :

Multiply the last $\left\{ \begin{array}{l} \text{Antecedent} \\ \text{Consequent} \end{array} \right\}$ by the next Quotient, and to the Product add the last $\left\{ \begin{array}{l} \text{Antecedent} \\ \text{Consequent} \end{array} \right\}$ but one; and so will the Sum be the present $\left\{ \begin{array}{l} \text{Antecedent.} \\ \text{Consequent.} \end{array} \right\}$

E X A M-

EXAMPLE.

Let it be required to find a Rank of Ratios, whose Terms are integral, and the nearest possible to the following Ratio, *viz.* of 10000 to 31416, which expresses nearly the Proportion of the Diameter of the Circle to its Circumference.

But because the Terms of the Ratio are not prime to each other, they must therefore be reduced to their least Terms.

Whence $\frac{10000}{31416} = \frac{1250}{3927}$, and then 3927 divided by 1250, and 1250 by the Remainder, &c. will be as follows:

$$\begin{array}{r} 1250 \) \ 3927 \ (\ 3 \\ \underline{177} \) \ 1250 \ (\ 7 \\ \underline{11} \) \ 177 \ (\ 16 \\ \underline{1} \) \ 11 \ (\ 11 \\ \underline{0} \end{array}$$

So the first Antecedent is 1, and the first Consequent 3.

$$\text{Lat } \left\{ \begin{array}{l} \text{Anteced. 1} \\ \text{Conseq. 3} \end{array} \right\} \times 7 = \left\{ \begin{array}{l} 7 \\ 21 \end{array} \right\} \text{ And } \left\{ \begin{array}{l} 7 + 0 = 7 \text{ the second Antec.} \\ 21 + 1 = 22 \text{ the second Conseq.} \end{array} \right.$$

Which 7 and 22 is *Archimedes's* Proportion.

$$\text{Lat } \left\{ \begin{array}{l} \text{Antec. 7} \\ \text{Conseq. 22} \end{array} \right\} \times 16 = \left\{ \begin{array}{l} 112 \\ 352 \end{array} \right\} \text{ And } \left\{ \begin{array}{l} 112 + 1 = 113 \text{ the 3d Antec.} \\ 352 + 3 = 355 \text{ the 3d Conseq.} \end{array} \right.$$

Which Terms 113 and 355 is *Metius's* Proportion.

$$\text{Lat } \left\{ \begin{array}{l} \text{Antec. 113} \\ \text{Conseq. 355} \end{array} \right\} \times 11 = \left\{ \begin{array}{l} 1243 \\ 3905 \end{array} \right\} \text{ And } \left\{ \begin{array}{l} 1243 + 7 = 1250 \\ 3905 + 22 = 3927 \end{array} \right.$$

Producing the same Antecedent and Consequent as at first; which, as it is ever the Property of the Rule so to do, proves, at the same Time, that no Error has been committed thro' the whole Operation.

$$\text{Whence, as } 1250:3927 :: \left\{ \begin{array}{l} 1:3 \\ 7:22 \\ 113:355 \end{array} \right\} \text{ For the } \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right\} \text{ Terms of the Ratio.}$$

But it must be observ'd, that 1 to 3 does not express the Ratio so near as 7 to 22, nor 7 to 22 so near as 113 to 355.

to 355 ; that is, the larger the Terms of the Ratio are, the nearer they approach the Ratio given.

Mr. *Molyneux*, in his Treatise of Dioptrics, informs us, that when Sir *Isaac Newton* set about, by Experiments, to determine the Ratio of the Angle of Incidence, to the refracted Angle, by the means of their respective Sines ; he found it to be from Air to Glass, as 300 to 193, or, in the least round Numbers, as 14 to 9. Now, if it be as 300 to 193, it will readily appear by the Rule, whether they are such integral Numbers, whose Ratio is the nearest possible to the given Ratio.

$$\begin{array}{r}
 193 \) \ 300 \ (\ 1 \\
 \underline{107} \) \ 193 \ (\ 1 \\
 \underline{86} \) \ 107 \ (\ 1 \\
 \underline{21} \) \ 88 \ (\ 4 \\
 \underline{2} \) \ 21 \ (\ 10 \\
 \underline{1} \) \ 2 \ (\ 2 \\
 \underline{0}
 \end{array}$$

For, dividing the greater Number by the less, and the less by the Remainder, &c. the Operation will shew, that the Numbers 193 and 300 are prime to each other ; and that the first Antecedent is 1, as also the first Consequent.

$$\begin{array}{l}
 \text{Whence } \left\{ \begin{array}{l} 1 \\ 1 \end{array} \right\} \times 1 = \left\{ \begin{array}{l} 1 \\ 1 \end{array} \right\} \text{ And } \left\{ \begin{array}{l} 1 \\ 1 \end{array} \right\} + 0 = 1 \text{ the second Antecedent.} \\
 \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right\} \times 1 = \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right\} \text{ And } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right\} + 1 = 2 \text{ the second Consequent.} \\
 \text{Again } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right\} \times 1 = \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right\} \text{ And } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right\} + 1 = 2 \text{ the third Antecedent.} \\
 \left\{ \begin{array}{l} 2 \\ 3 \end{array} \right\} \times 1 = \left\{ \begin{array}{l} 2 \\ 3 \end{array} \right\} \text{ And } \left\{ \begin{array}{l} 2 \\ 3 \end{array} \right\} + 2 = 3 \text{ the third Consequent.} \\
 \text{Again } \left\{ \begin{array}{l} 2 \\ 3 \end{array} \right\} \times 4 = \left\{ \begin{array}{l} 8 \\ 12 \end{array} \right\} \text{ And } \left\{ \begin{array}{l} 8 \\ 12 \end{array} \right\} + 1 = 9 \text{ the fourth Antecedent.} \\
 \left\{ \begin{array}{l} 8 \\ 12 \end{array} \right\} \times 2 = \left\{ \begin{array}{l} 16 \\ 24 \end{array} \right\} \text{ And } \left\{ \begin{array}{l} 16 \\ 24 \end{array} \right\} + 2 = 14 \text{ the fourth Consequent.}
 \end{array}$$

Hence, the fourth Antecedent and Consequent make the Ratio to be as 9 to 14, or, inverfly, as 14 to 9 ; which not only agrees with Mr. *Molyneux*, but at the same time discovers, that they are nearer to the given Ratio, than any other integral Numbers less than 92 and 143 ; which are the nearest of all to the given Ratio, as will appear by repeating the Process, according to the Direction of the Rule.

Sir *Isaac Newton* himself determines the Ratio out of Air into Glass to be as 17 to 11 ; but then he speaks of the red Light. For that great Philosopher, in his

Dissertations concerning Light and Colours, publish'd in the *Philosophical Transactions*, has at large demonstrated, as also in his Optics, that the Rays of Light are not all homogeneous, or of the same sort, but of different Forms and Figures, so that some are more refracted than others, tho' they have the same or equal Inclinations on the Glass: Whence there can be no constant Proportion settled between the Sines of the Incidence, and of the refracted Angles.

But the Proportion that comes nearest Truth, for the middle and strong Rays of Light, it seems, is nearly as 300 to 193, or 14 to 9. In Light of other Colours the Sines have other Proportions. But the Difference is so little, that it need seldom to be regarded, and either of those mention'd for the most part is sufficient for Practice. However, I must observe, that the Notice here taken either of the one or the other, is more to illustrate the Rule, and shew, as Occasion requires, how to express any given Ratio in smaller Terms, and the nearest possible, with more Ease and Certainty, than any Design in the least of touching upon Optics.

Wherefore, lest this small Digression from the Subject in hand, and indeed even from my first Intentions, should tire the Reader's Patience, I shall not presume more, but immediately proceed to the Construction of Logarithms.

Of the Construction of Logarithms.

THE Nature of which, tho' our Author has sufficiently explain'd in the Description of the Logarithmical Curve; yet, before we attempt their Construction, it will be necessary to premise:

That the Logarithm of any Number is the Exponent or Value of the Ratio of Unity to that Number; wherein we consider Ratio, quite different from that laid down in the fifth Definition of the 5th Book of these Elements; for, beginning with the Ratio of Equality, we say 1 to 1 = 0, whereas, according to the said Definition, the Ratio of 1 to 1 = 1; and consequently the Ratio here mention'd is of a peculiar Nature, being affirmative when increasing, as of Unity to a greater Number; but negative when decreasing. And

as the Value of the Ratio of Unity to any Number is the Logarithm of the Ratio of Unity to that Number, so each Ratio is supposed to be measured by the Number of equal Ratiunculæ contained between the two Terms thereof: Whence, if in a continued Scale of mean Proportionals, infinite in Number, there be assumed an infinite Number of such Ratiunculæ, between any two Terms in the said Scale; then that infinite Number of Ratiunculæ is to another infinite Number of the like and equal Ratiunculæ between any other two Terms, as the Logarithm of the one Ratio is to the Logarithm of the other.

But if, instead of supposing the Logarithms composed of a Number of equal Ratiunculæ proportionable to each Ratio, we shall take the Ratio of Unity to any Number to consist always of the same infinite Number of Ratiunculæ, their Magnitudes in this Case will be as their Number in the former. Wherefore, if between Unity and any two Numbers proposed, there be taken any Infinity of mean Proportionals, the infinitely little Augments or Decrements of the first of those Means in each from Unity will be Ratiunculæ; that is, they will be the Fluxions of the Ratio of Unity to the said Numbers; and because the Number of Ratiunculæ in both are equal, their respective Sums, or whole Ratios, will be to each other as their Moments or Fluxions; that is, the Logarithms of each Ratio will be as the Fluxion thereof. Consequently, if the Root of any infinite Power be extracted out of any Number, the Difference of the said Root from Unity shall be as the Logarithm of that Number. So that Logarithms thus produced may be of as many Forms as we please to assume infinite Indices of the Power whose Root we seek. As, if the Index be supposed 100000, &c. we shall have the Logarithms invented by *Napier*; but if the said Index be 230258, &c. those of Mr. *Briggs's* will be produced.

Wherefore, if $1+x$ be any Number whatsoever, and n infinite, then its Logarithm will be as $\overline{1+x^n} - 1 = \frac{1}{n} \times x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}, \&c.$ For the infinite Root of $1+x$ without its Unciæ, or prefix'd Numbers,

is

is $1+x+xx+xxx+xxxx$, &c. and the celebrated binomial Theorem invented by Sir *Isaac Newton* for

determining them is $1 \times \frac{1}{n} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, &c. or

in this Case rather $1 \times \frac{1}{n} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, &c. for $\frac{1}{n}$ being an Infinitesimal, is rejected; whence the infi-

nite Root of $1+x = 1 + \frac{x}{n} + \frac{xx}{2n} + \frac{x^3}{3n} + \frac{x^4}{4n}$

$\frac{x^5}{5n}$, &c. and the Excess thereof above Unity, viz. $\frac{x}{n} -$

$\frac{xx}{2n} + \frac{x^3}{3n} - \frac{x^4}{4n}$, &c. is the Augment of the first of the

mean Proportionals between Unity and $1+x$, which therefore will be as the Logarithm of the Ratio of 1 to

$1+x$, or as the Logarithm of $1+x$. But as $1+x-1$ is a Ratiuncula, it must be multiplied by 10000, &c. in-

finitely, which will reduce it to Terms fit for Practice, making the Logarithm of the Ratio of 1 to $1+x =$

$\frac{1000}{n} \times \frac{x}{1} - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, &c. whence if the

Index n be taken 1000, &c. as in *Napier's* Form, the

Logarithms will be simply $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$, &c.

But as n may be taken at Pleasure, the several Scales of 1000, &c.

Logarithms to such Indices will be as $\frac{1}{n}$, or reciprocally as their Indices.

Again, if the Logarithm of a decreasing Ratio be

sought, the infinite Root of $1-x = 1 - \frac{x}{n}$ will be

found by the like Method to be $1 - \frac{x}{n} + \frac{1}{2n} x^2 - \frac{1}{3n} x^3 +$

$\frac{1}{4n} x^4$, &c. which subtract from Unity, and the

Decrement of the first of the infinite Number of Proportionals will appear to be $\frac{1}{n} x + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \frac{1}{4} x^4$

x^4 , &c. which expresses the Logarithm of the Ratio of 1 to $1-x$, or the Logarithm of $1-x$ according to Napier's Form, if the Index n be put = 10000, &c. as before.

And to find the Logarithm of the Ratio of any two Terms, a the least, and b the greater, it will be, as $a : b :: 1 : 1+x : x = \frac{b}{a} - 1 = \frac{b-a}{a}$, or the Difference divided by the lesser Term when 'tis an increasing Ratio, and $\frac{b-a}{b}$ when 'tis decreasing.

Wherefore, putting d = Difference between the two Terms a and b , the Logarithms of the same Ratio may be doubly expressed, and accordingly is either $\frac{1}{n} \times \frac{d}{a} - \frac{d^2}{2a^2} + \frac{d^3}{3a^3} - \frac{d^4}{4a^4}$, &c. or $\frac{1}{n} \times \frac{d}{b} + \frac{d^2}{2b^2} + \frac{d^3}{3b^3} + \frac{d^4}{4b^4}$, &c. both producing the same Thing.

But if the Ratio of a to b be supposed to be divided into two Parts, viz. into the Ratio of a to the arithmetical Mean between the two Terms, and the Ratio of the said arithmetical Mean to the other Term b , then will the Sum of the Logarithms of those two Ratios be the Logarithm of the Ratio of a to b . Wherefore substituting $\frac{1}{2}s$ for $\frac{1}{2}a + \frac{1}{2}b$, and it will be, $\frac{1}{2}s : a :: 1 : 1-x$; whence $x = \frac{\frac{1}{2}s - a}{a} = \frac{\frac{1}{2}s - a}{a}$. And again, as

$\frac{1}{2}s : b :: 1 : 1+x$; whence $x = \frac{b - \frac{1}{2}s}{b} = \frac{b - \frac{1}{2}s}{b}$. Therefore substituting $\frac{s}{2}$ for x , and we shall have for both Ratios

$\frac{1}{n} \times \frac{d}{s} + \frac{d^2}{2s^2} + \frac{d^3}{3s^3} + \frac{d^4}{4s^4}$, &c. $\frac{1}{n} \times \frac{d}{s} - \frac{d^2}{2s^2} + \frac{d^3}{3s^3} - \frac{d^4}{4s^4}$, &c. And their Sum, viz. $\frac{1}{n} \times 2 \times \frac{d}{s} + \frac{d^3}{3s^3} + \frac{d^5}{5s^5} + \frac{d^7}{7s^7}$, &c. is the Logarithm of the Ratio of a to b , whose Difference is d , and Sum s ; which Series, without the Index n , is, by-the-bye, the

6

Fluent

Fluent of the Fluxion of the Logarithm of $\frac{s+d}{s-d}$, assuming d , the flowing Quantity, for the Fluxion of the Log. of $\frac{s+d}{s-d}$, is $\frac{2s\dot{d}}{ss-dd} = 2 \times \frac{\dot{d}}{s} + \frac{d^2\dot{d}}{s^3} + \frac{d^4\dot{d}}{s^5} + \frac{d^6\dot{d}}{s^7} + \text{etc.}$ whose Fluent $2 \times \frac{d}{s} + \frac{d^3}{3s^3} + \frac{d^5}{5s^5} + \frac{d^7}{7s^7} + \text{etc.}$

is *Napier's* Logarithm of $\frac{s+d}{s-d}$, and the same as above,

bating the Index n . This Series, either way obtain'd, converges twice as swift as the former, and consequently is more proper for the Practice of making Logarithms:

Thus, put $a=1$, and b any Number at Pleasure; then $\frac{d}{s} = \frac{b-1}{b+1}$, which assume $=e$, and then $b = \frac{e+1}{1-e}$; and

because $\frac{d}{s} = e$, therefore we have for

THEOREM I.

The Log. of $b (= \frac{e+1}{1-e}) = \frac{1}{n} \times 2 \times e + \frac{1}{3}e^3 + \frac{1}{5}e^5 + \frac{1}{7}e^7 + \text{etc.}$

To illustrate this Theorem: Let it be required to find the Logarithm of 2 true to 7 Places.

Note, That the Index must be assumed of a Figure or two more than the intended Logarithm is to have.

EXAMPLE.

Here $b = \frac{e+1}{1-e} = 2 \dots e+1 = 2 - 2e \dots 3e = 1$; whence $e = \frac{1}{3}$, and $ee = \frac{1}{9}$.

The OPERATION stands thus :

$\frac{1}{3}=e = ,33333333$	$e = ,33333333$
$e^3 = 3703704$	$\frac{1}{3}e^3 = 1234568$
$e^5 = 411523$	$\frac{1}{5}e^5 = 82304$
$e^7 = 45725$	$\frac{1}{7}e^7 = 6532$
$e^9 = 5080$	$\frac{1}{9}e^9 = 564$
$e^{11} = 564$	$\frac{1}{11}e^{11} = 51$
$e^{13} = 63$	$\frac{1}{13}e^{13} = 5$
	$,34657357$
	$\quad\quad\quad 2$

Whence *Napier's* Logarithm of 2 is $,69314714$
 But $,69314714$, multiplied by 3, will give 207944142
 for the Logarithm of 8, inasmuch as 8 is the Cube or
 third Power of 2, and the Logarithm of 8 plus the Log.
 of $1\frac{1}{4}$ is equal to the Logarithm of 10, because 8
 $\times 1\frac{1}{4} = 10$; wherefore to find the Logarithm of $1\frac{1}{4}$

$$\text{we have } b = \frac{e+1}{1-e} = \frac{1}{\frac{1}{2}} = \frac{1}{2}; \text{ whence } e = \frac{1}{9}, \text{ and}$$

$$ee = \frac{1}{81}.$$

The OPERATION stands thus :

$\frac{1}{9}=e = ,11111111$	$e = ,11111111$
$e^3 = 137172$	$\frac{1}{3}e^3 = 45724$
$e^5 = 1693$	$\frac{1}{5}e^5 = 338$
$e^7 = 21$	$\frac{1}{7}e^7 = 3$
	$,11157176$
	$\quad\quad\quad 2$

Whence *Napier's* Logarithm of $1\frac{1}{4}$ is $,22314352$
 To which add the Logarithm of 8, $2,07944142$
 The Sum, viz. $2,30258494$
 is *Napier's* Logarithm of 10. But if the Logarithm of
 10 be made 1,000000, &c. as it is most conveniently
 done in most of our Tables extant, then $\frac{2302585}{n} =$

1,000, &c. Whence $n=2302585$, &c. is the Index
 for *Briggs's* Scale of Logarithms; and, if the above
 Work had been carried on to Places sufficient, the Index
 n would have been 2,30258, 50929, 94045, 68401,
 79914,

79914, &c. and its Reciprocal, viz. $\frac{1}{n} = 0,43429,$
 44819, 03251, 82765, 11289, &c. which, by the way,
 is the Subtangent of the Curve expressing Briggs's
 Logarithms; from the Double of which the said Loga-
 rithms may be had directly.

For, because $\frac{1}{n} = 0,4342944, \&c. \therefore \frac{2}{n} = ,868588$
 9638, &c. which put $= m$, and then the Logarithm
 of $b = \frac{e+1}{1-e} = me + \frac{me^3}{3} + \frac{me^5}{5} + \frac{me^7}{7} + \frac{me^9}{9}, \&c.$

EXAMPLE.

Let it be required to find Briggs's Logarithm of 2.

Here $b = \frac{e+1}{1-e} = 2 \therefore e = \frac{1}{3}$, and $ee = \frac{1}{9}$.

The OPERATION stands thus:

m	$= ,868588963$		me	$= ,289529655$
me	$= ,289529655$		$\frac{1}{3}me^3$	$= 10723321$
me^3	$= 32169962$		$\frac{1}{3}me^5$	$= 714888$
me^5	$= 3574440$		$\frac{1}{7}me^7$	$= 56737$
me^7	$= 397160$		$\frac{1}{9}me^9$	$= 4902$
me^9	$= 44120$		$\frac{1}{11}me^{11}$	$= 445$
me^{11}	$= 4902$		$\frac{1}{13}me^{13}$	$= 42$
me^{13}	$= 545$			

Whence Briggs's Logarithm of 2 is 0,30102999.

A G A I N:

Let it be requir'd to find Briggs's Logarithm of 3.
 Now because the Logarithm of 3 is equal to the Loga-
 rithm of 2 plus the Logarithm of $1\frac{1}{2}$ (for $2 \times 1\frac{1}{2} =$
 3), therefore find the Logarithm of $1\frac{1}{2}$, and add it to
 the Logarithm of 2 already found, the Sum will be the
 Logarithm of 3, which is better than finding the Lo-
 garithm of 3 by the Theorem directly, inasmuch as
 it will not converge so fast as the Logarithm of $1\frac{1}{2}$; for
 the smaller the Fraction represented by e , which is

deduc'd from the No. whose Logarithm is sought, the swifter does the Series converge.

$$\text{Here } b = \frac{e+1}{1-e} = \frac{1}{2} \cdot \cdot \cdot 2e+2 = 3-3e \cdot \cdot \cdot e = \frac{1}{3},$$

$$\text{and } ee = \frac{1}{2},$$

The OPERATION is as follows:

m	$=,868588063$		me	$=,173717792$
me	$=,173717792$		$\frac{1}{2}me$	$=,086858896$
me^3	$=,005248071$		$\frac{1}{2}me^3$	$=,002624035$
me^5	$=,000152063$		$\frac{1}{2}me^5$	$=,000076031$
me^7	$=,000004167$		$\frac{1}{2}me^7$	$=,000002083$
me^9	$=,000000117$		$\frac{1}{2}me^9$	$=,000000058$
me^{11}	$=,000000003$		$\frac{1}{2}me^{11}$	$=,000000001$

$$\text{Briggs's Logarithm of } 1\frac{1}{2} = ,176091256$$

$$\text{To which add the Logarithm of } 2 = ,301029990$$

$$\text{The Sum is the Logarithm of } 3 = ,477121246$$

Again, to find the Logarithm of 4, because $2 \times 2 = 4$, therefore the Logarithm of 2 added to itself, or multiplied by 2, the Product 0,60205998 is the Logarithm of 4.

To find the Logarithm of 5, because $\frac{1}{2}^2 = 5$, therefore from the Logarithm of 10 1,000000000 subtract the Logarithm of 2

$$\text{There remains the Logarithm of } 5 = ,698970010$$

And because $2 \times 3 = 6$; therefore

To find the Logarithm of 6,

$$\text{To the Logarithm of } 3 = ,477121246$$

$$\text{Add the Logarithm of } 2 = ,301029990$$

$$\text{The Sum will be the Logarithm of } 6 = ,778151236$$

Which being known, the Logarithm of 7, the next prime Number, may be easily found by the Theorem; for because $6 \times \frac{7}{6} = 7$, therefore to the Logarithm of 6 add the Logarithm of $\frac{7}{6}$, and the Sum will be the Logarithm of 7.

EXAMPLE.

$$\text{Here } b = \frac{e+1}{2-e} = \frac{2}{3} \cdot \cdot \cdot e = \frac{1}{3} \text{ and } ee = \frac{1}{9}.$$

$$m =$$

$m = ,868588963$		$me = ,066814535$
$me = ,066814535$		$\frac{1}{3}me^3 = 131784$
$me^3 = 395352$		$\frac{1}{5}me^5 = 467$
$me^5 = 2339$		$\frac{1}{7}me^7 = 2$
$me^7 = 13$		

Briggs's Logarithm of 2 ,066946788
 To which add the Log. of 6 ,778151236
 The Sum is the Log. of 7 = ,845098024

Again, because $4 \times 2 = 8$; therefore
 To the Logarithm of 4 ,60205998
 Add the Logarithm of 2 ,30102999
 The Sum is the Logarithm of 8 ,90308997

And because $3 \times 3 = 9$; therefore
 To the Logarithm of 3 ,477121246
 Add the Logarithm of 3 ,477121246
 The Sum is the Logarithm of 9 ,954242492

And the Logarithm of 10 having been determined to be 1,0000000, we have therefore obtained the Logarithms of the first ten Numbers.

After the same manner the whole Table may be constructed, and as the prime Numbers increase, so fewer Terms of the Theorem are requir'd to form their Logarithms; for in the common Tables, which extend but to seven Places, the first Term is sufficient to produce the Logarithm of 101, which is compos'd of the Sum of the Logarithms of 100 and $\frac{101}{100}$, because 100

$\times \frac{101}{100} = 101$, in which Case $b = \frac{e+1}{1-e} = \frac{101}{100} \dots e =$

$\frac{1}{20}$; whence, in making of Logarithms according to the preceding Method, it may be observ'd, that the Sum and Difference of the Numerator and Denominator of the Fraction whose Logarithm is sought, is ever equal to the Numerator and Denominator of the Fraction represented by e ; that is, the Sum is the Denominator, and the Difference, which is always Unity, is the Numerator; consequently the Logarithm of any prime Number may be readily had by the Theorem, having the Logarithm either next above or below given.

Tho', if the Logarithms next above and below that Prime are both given, then its Logarithm will be obtained something easier. For half the Difference of the Ratios

Ratios which constitute the 1st Theorem, viz. ($n =$)
 $m \times \frac{dd}{2ss} + \frac{d^2}{4s^2} + \frac{d^6}{6s^6} + \frac{d^8}{8s^8}$, &c. is the Logarithm of
 the Ratio of the arithmetical Mean to the geometrical
 Mean, which being added to the half Sum of the Lo-
 garithms next above and below the Prime sought, will
 give the Logarithm of that prime Number, which for
 Distinction's sake may be call'd Theorem the second,
 and is of good Dispatch, as will appear hereafter by an
 Example.

But the best for this Purpose is the following one,
 which is likewise deriv'd from the same Ratios as Theo-
 rem the first. For the Difference of the Terms be-
 tween ab and $\frac{1}{4}ss$, or $\frac{1}{4}aa + \frac{1}{2}ab + \frac{1}{4}bb$, is $\frac{1}{4}aa$

$-\frac{1}{2}ab + \frac{1}{4}bb = \frac{1}{4}a - \frac{1}{2}b = \frac{1}{4}dd = 1$, and the Sum
 of the Terms ab and $\frac{1}{4}ss$ being put $= y$, therefore
 (since y in this Case $= s$, and $d = 1$) it follows that

$\frac{1}{n} \times \frac{1}{y} + \frac{1}{y^2} + \frac{1}{y^3} + \frac{1}{y^4}$, &c. is the Logarithm of the

Ratio of ab to $\frac{1}{4}zz$: Whence $\frac{1}{n} \times \frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^5}$

$\frac{1}{7y^7}$, &c. is the Logarithm of the Ratio of $\frac{1}{2}s$ to $\sqrt{ab}$,
 $7y$

which converges exceeding quick, and is of excellent
 Use for finding the Logarithm of prime Numbers,
 having the Logarithms of the Numbers next above
 and below given, as in Theorem the second.

EXAMPLE.

Let it be requir'd to find the Logarithms of the prime
 Number 101; then $a = 100$, and $b = 101$; whence

$y = 20401$; put $\frac{1}{n} = m = .4342944819$, &c. Then

the Series will stand thus, $\frac{m}{y} + \frac{m}{3y^3} + \frac{m}{5y^5} + \frac{m}{7y^7}$, &c.

And

And $m=,43429$, &c. divided by } ,0000212879014
 $y=20401$, quotes
 Therefore to the half Sum of the } 2,0043000858809
 Logarithms of 100 and 102 =
 Add the said Quote 0,20000212879014
 And the Sum, viz. 2,0043213737823

is the Logarithm of 101 true to 12 Places of Figures, and obtained by the first Term of the Series only; whence 'tis easy to perceive what a vast Advantage the second Term would have, were it put in Practice, since m is to be divided by 3 multiplied into the Cube of 20401.

This Theorem, which we'll call Theorem the third, was first found out by Dr. *Halley*, and a notable Instance of its Use given by him in the *Philosophical Transactions* for making the Logarithm of 23 to 32 Places, by five Divisions performed with small Divisors; which could not be obtained according to the Methods first made use of, without indefatigable Pains and Labour, if at all; on account of the great Difficulty that would attend the managing such large Numbers.

Our Author's Series for this Purpose is (Page 363.)

$y \times \frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5}$, &c. the Investigation of which as he was pleased to conceal, induced me to inquire into it, as well to know the Truth of the Series, as to know whether this or that had the Advantage, because Dr. *Halley* informs us, when his was first published, that it converged quicker than any Theorem then made public, and in all Probability does so still; however that be, 'tis certain our Author's converges no faster than the second Theorem, as I found by the Investigation thereof, which may be as follows:

From the foregoing Doctrine the Difference of the Logarithms of $z-1$ and $z+1$ is $m \times \frac{2}{z} + \frac{2}{3z^3} + \frac{2}{5z^5} + \frac{2}{7z^7} + \frac{2}{9z^9}$, &c. which put equal to y , and the Logarithm of the Ratio of the Arithmetical Mean z , to the Geometrical Mean $\sqrt{z^2-1}$ is $m \times \frac{1}{2zz} + \frac{1}{4z^4} + \frac{1}{6z^6} + \frac{1}{8z^8}$, &c. per Theorem the second:

for

for z equal to $\frac{1}{2} s$; whence $\frac{dd}{2ss} = \frac{1}{2zz}$.

Let A and B be the Logarithms of $z-1$ and $z+1$ respectively; then is $\frac{A+B}{2} + m \times \frac{1}{2zz} + \frac{1}{4z^3} + \frac{1}{6z^5}$

$+ \frac{1}{8z^7}, \&c.$ the Logarithm of z , and if the latter Part

of the Series expressing the said Logarithm of z be divided by the Series representing the Difference of the Logarithms of $z-1$ and $z+1$, the Quotient will exhibit the Series required, *viz.* $\frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5}, \&c.$

as appears by the following Operation:

$$\begin{array}{r} \frac{z}{z} + \frac{2}{3z^3} + \frac{2}{5z^5}, \&c. \quad \left(\frac{1}{2z^2} + \frac{1}{4z^4} + \frac{1}{6z^6}, \&c. \right) \left(\frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5}, \&c. \right) \\ \hline \frac{1}{2z^2} + \frac{1}{6z^4} + \frac{1}{10z^6}, \&c. \\ \hline \frac{1}{12z^4} + \frac{1}{15z^6}, \&c. \\ \hline \frac{1}{12z^4} + \frac{1}{36z^6}, \&c. \\ \hline \frac{1}{180z^6}, \&c. \end{array}$$

Now, because the Dividend is ever equal to the Divisor drawn into the Quotient of the Division, it follows,

that $y \times \frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5}, \&c.$ is equal to $m \times \frac{1}{2zz}$

$+ \frac{1}{4z^3} + \frac{1}{6z^5}, \&c.$ But $\frac{A+B}{2} + m \times \frac{1}{2zz} + \frac{1}{4z^3}$

$+ \frac{1}{6z^5}, \&c.$ is the Logarithm of z . Wherefore $\frac{A+B}{2}$

$+ y \times \frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5}, \&c.$ is the Logarithm of

z . Q. E. I.

Note, I make the Author's 5th Term $\frac{13}{25200z^9}$

to be $\frac{1903}{226800z^9}$.

To illustrate this Theorem by an Example :

Let it be required to find the Logarithm of 101.

To the half Sum of the Logarithms of 101 and 102 =

2,0043000858

Add the Difference of the said Logarithms divided by 42 equal to } 0,0000212875

And the Sum, viz. - - - - - 2,0043213733

is the Logarithm of 101 true to 9 Places of Figures : Whence it appears, that our Author's Series falls short of Dr. Halley's in finding the Logarithm of the prime Number 101, three Places of Figures, by using only the first Terms of the Series ; whereas, if two Terms in each were used, perhaps the Difference would have been considerably greater.

Note, This Series of our Author, deduc'd from Theorem the Second, is in Effect Dr. Halley's too, but veil'd over by being thrown into a different Form ; which, however, has its Use, as being very ready in Practice.

Having thus investigated several Theorems, whereby the Tables of Logarithms, according to any Form, may be constructed ; it remains to shew how, from the Logarithm given, to find what Ratio it expresses.

The Logarithm of the Ratio of 1 to $1+x$ has been

$\frac{1}{n}$
prov'd to be as $1+x - 1 = \frac{1}{n} \times x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4$,
&c. n being any infinite Index whatsoever ; whence, if

$\frac{1}{n}$
L be put for the said Series, then $1+x - 1 = L$; con-

$\frac{1}{n}$ frequently $1+x = 1+L$, and $1+x = 1+L = 1 + nL + \frac{1}{2} n^2 L^2 + \frac{1}{6} n^3 L^3 + \frac{1}{24} n^4 L^4$, &c.

A G A I N :

The Logarithm of the Ratio of 1 to $1-x$ has like-

$\frac{1}{n}$
wise been prov'd to be as $1-1-x = n \times x + \frac{1}{2} x^2 +$

$\frac{1}{6} x^3 + \frac{1}{24} x^4$, &c. = L ; wherefore $1-x = 1-L$,
and

and $1-x = \frac{n}{1-L} = 1-nL + \frac{1}{2}n^2L^2 - \frac{1}{6}n^3L^3 + \frac{1}{24}n^4L^4, \text{ \&c.}$

Whence $1+x = 1+nL + \frac{1}{2}n^2L^2 + \frac{1}{6}n^3L^3 + \frac{1}{24}n^4L^4, \text{ \&c.}$ is a general Theorem for finding the Number from the Logarithm given of any Species or Form whatsoever; but in the Application of it to Practice we labour under a great Inconveniency, especially if the Numbers are large; that is to say, it converges so very slow, that it were much to be wished it could be contracted.

However, if L be the Logarithm of the Ratio of a the lesser Term, to b the greater, and either of them are given; then the other will be easily had, and expeditiously enough too:

For $b \left\{ = \begin{matrix} a \\ b \end{matrix} \right\} \times 1 + nL + \frac{1}{2}n^2L^2 + \frac{1}{6}n^3L^3, \text{ \&c.}$
and a

Wherefore it follows, by the Help of a Table of Logarithms, that the corresponding Number to any Logarithm may be found to as many Places of Figures as those Logarithms consist of: For, putting d equal to the Difference between the given Logarithm and the next less in the Table, then will the Number sought, viz. $N = a \times 1 + nd + \frac{1}{2}n^2d^2 + \frac{1}{6}n^3d^3, \text{ \&c.}$ But if d be put equal to the Difference between the given Logarithm, and the next greater, then $N = b \times 1 - nd + \frac{1}{2}n^2d^2 - \frac{1}{6}n^3d^3, \text{ \&c.}$ both of which Series converge faster, as d is smaller.

But the first three Terms in each may be contracted into two, which is very useful, inasmuch as it saves the Trouble of raising n and d in the third Term to the second Power: For letting the first Term remain as it is, the other two are reduc'd to one, thus; make the second Term the Numerator of a Fraction, and Unity minus the third Term divided by the second is the Denominator.

Whence $N = a \times 1 + nd + \frac{1}{2}n^2d^2 = a + \frac{a d}{1 - \frac{1}{2}d}$; and $N = b \times d^2 - nd + \frac{1}{2}n^2 = b - \frac{b d}{\frac{1}{2} + \frac{1}{2}d}$.

Wherefore $a + \frac{a d}{m - \frac{1}{2}d}$, or $b - \frac{b d}{m + \frac{1}{2}d}$, will be the
Num-

Number answering to the given Logarithm, which tho' it differs a little from the Truth, is sufficient to find the Numbers exact to as many Places as *Briggs's* Logarithms consist of, viz. 14, which are the largest Tables extant. Much after the same Method may the whole Series be contracted, by which means each alternate Power of d will be exterminated, or, which is the same thing, every two Terms in the Series will be reduc'd to one, making the whole as short again.

To illustrate these Contractions by an Example:

Let it be required to find the Number answering to the Logarithm 7,5713740282 in *Briggs's* Form.

From the given Logarithm 7,5713740282
Subtract the Log. of 372710000 the }
next nearest, 7,5713710453

The Remainder is equal to $d =$,0000029829

And because the Number 372710000 is less than the Number sought, call it a , which, multiplied by ,0000029829, and the Product 1,11175 6659, &c. divided by $m - \frac{1}{2}d =$,434299, &c. quotes ,255,992; which, added to 37271000, gives 37271255,992 for the Number sought.

Thus, I presume, the Doctrine of Logarithms has been sufficiently exemplified, whether we consider the Construction of them for any given Numbers; or, on the contrary, the finding of the Numbers from the Logarithms given.

But, before I conclude, I shall give an Instance or two of the great Use of Logarithms in Arithmetical Calculations, and first in the purchasing of Annuities.

If a be put for any Annuity, p for the present Value, r the Amount of one Pound for one Year at any Rate of Interest, and t for the Time or Number of

Years the Annuity is to continue, then, $p = \frac{a - \frac{a}{r^t}}{r - 1}$ the Value of the Annuity.

EXAMPLE.

Let it be required to find the present Value of an Annuity of 60 *l.* per Annum to continue 75 Years at the Rate of 4 per Cent. per Annum.

Here

Here $a=60$, $t=75$, and $r=1.04$. Now, in order to obtain the Answer, we must find the seventy-fifth Power of r , or of 1.04 ; that is, we must multiply 1.04 seventy-five times into itself, which is exceeding tedious by the common way, as any one may judge; but by the Logarithms 'tis done with the greatest Ease; for if 0.017033 the Logarithm of 1.04 be multiplied by 75 , the Product 1.277475 will be the Logarithm of the seventy-fifth Power of 1.04 ; which being subtracted from 1.778151 , the Logarithm of a equal to 60 , will leave 0.500676 the Logarithm of 3.16719 , which being subtracted from 60 , and the Remainder divided by $r-1=.04$ will give 1421.820 equal to $1421.15s.4d.\frac{3}{4}$ for the Value of the Annuity; and if 1421.820 be divided by 60 , the Quotient will exhibit the Number of Years Purchase requisite to be given for any Annuity to continue 75 Years upon a good Security free of all Incumbrances, the Purchase being made at 4 per Cent.

Hence we see the Reason why the long Annuities purchased in the Year 1708 , having about 75 Years to come, are now valued in *Castain's* Bill of Exchanges at $24\frac{1}{2}$ or 25 Years Purchase; for tho', according to this Calculation, they are worth but a little more than 23 Years and a half; yet, because in the public Funds 4 per Cent. is scarcely ever made of Money, and the Contingencies it is there subject to, which those Annuities and other Government Securities are not, makes them very justly worth $24\frac{1}{2}$ or 25 Years Purchase.

Likewise Questions relating to Annuities upon Lives, whether for one, two, or three, &c. are almost as easily estimated. For Instance, it may readily be found by Logarithms, that an Annuity for a Man of Thirty to continue during his Life is worth 11.61 Years Purchase, Interest 6 per Cent. but at 4 per Cent. 14.68 . And as the Probabilities of Life's Continuance, and the Value thereof, are determined by an algebraical Process grounded upon the Rudiments of the Doctrine of Chances, and five Years Observations upon the Bills of Mortality of *Breslaw*, the Capital of *Silesia*; so there results that Truth and Equity from the Operations, as ought to preside in all Contracts of this Nature. Whence it follows, that all other Methods, whose Resolution differs from this, (especially if the Difference be much) may justly be deem'd

deem'd erroneous, consequently prejudicial to one of the Parties concern'd. Wherefore, to prevent Impositions thro' Ignorance, great Care should be taken; which Precaution, however unnecessary it may appear, 'tis presum'd, will be regarded, inasmuch as no one is willing to pay more Years Purchase than he has Chances for living, as, on the contrary, the Seller to receive less than his Due, which may possibly be by following the common Methods, where, for the most part, Regard is had neither to Age nor Interest, but founded upon Caprice, Humour, or, if you please, Custom, the Contract being made, as they can agree, right or wrong; which Method of Procedure ought to be exploded, since so liable to Error, and the Consequences drawn therefrom, so often wide of the Truth.

The other Instance which I shall give of the great Use of Logarithms is in the Case of *Sessa*, as related by Dr. *Wallis* in his *Opus Arithmeticum* from *Alsephad* (an *Arabian* Writer) in his Commentaries upon *Tograius's* Verses, namely, that one *Sessa*, an *Indian*, having first found out the Game at *Chesse*, and shew'd it to his Prince *Shebram*, the King, who was highly pleas'd with it, bid him ask what he would for the Reward of his Invention; whereupon he ask'd, That for the first little Square of the *Chesse* Board he might have one Grain of Wheat given, for the second two, and so on, doubling continually, according to the Number of the Squares in the *Chesse* Board, which was 64. And when the King, who intended to give a very noble Reward, was much displeas'd, that he had ask'd so trifling a one, *Sessa* declar'd, that he would be contented with this small one. So the Reward he had fix'd upon was order'd to be given him. But the King was quickly astonish'd, when he found, that this would rise to so vast a Quantity, that the whole Earth itself could not furnish out so much Wheat. But how great the Number of these Grains is, may be found by doubling one continually 63 times, so that we may get the Number that comes in the last Place, and then one time more to have the Sum of all; for the Double of the last Term less by one is the Sum of all. Now this will be more expeditiously done by Logarithms, and accurately enough too for this Purpose: For if to the Logarithm of 1, which is 0, we add the Logarithm of 2, which is

0,3010300 multiplied by 64, that is, 19,2659200, the absolute Number agreeing to this will be greater than 18466, 00000, 00000, 00000, and less than 18447, 00000, 00000, 00000.

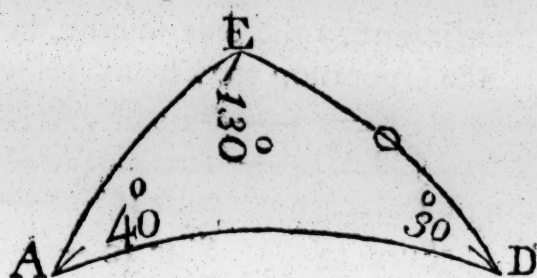
As I have had the revising of these Sheets, so it may be expected that I should give my Opinion concerning Mr. *Cunn* and our Author, in regard to Spherical Trigonometry; wherein the former accuses the latter, and several other eminent Authors, of having committed many Faults, and, in some Cases, of being mistaken, especially in the Solution of the 12th Case of Oblique Spherics; in which Mr. *Cunn* has intirely mistook the Author's Meaning, as plainly appears by his Remark, where he constitutes a Triangle whose Sides are equal to the given Angles; whereas the Author means, that each Angle should first be chang'd into its Supplement, and then with the said Supplement another Triangle constituted, whose Angles, by the very Text of the 14th Proposition of his own Spherical Trigonometry, will be the Supplements of the Sides sought in the given Triangle; to which Proposition I refer the Reader. That this is the Sense of the Author, is very evident, if impartially attended to, and which I think could possibly have no other Meaning; and accordingly aver what is here advanc'd to be universally true; but because I would not be misunderstood, shall illustrate the Truth thereof by a numerical Operation; which, to those who care not to trouble themselves with the Demonstration, may be sufficient; and, to others, some Satisfaction.

E X A M P L E.

Suppose, in the Oblique-angled Spherical Triangle ADE, there are given the Angles A, D, E, as *per* Figure, and the Side DE requir'd.

Note, Write down the Supplements of the two Angles next the Side requir'd first; and then the Operation may stand thus:

The



The Supplement of the Angle	{	E = 50°	Sine Co. Ar.	—	0,115746
		D = 150	Sine Co. Ar.	—	0,301030
		A = 140	Sine 120°	—	9,937530
		Sum = 340	Sine 20	—	9,535052
		2			—
		1/2 Sum = 170			19,884358
				2	—

1/2 Sum, minus the Supplement of the Angle	{	E = 120			
		D = 20			9,944179

Which last Figures, 9,944179, give the Sine of $61^{\circ} 34'$; and the Double thereof, viz. $123^{\circ} 08'$, subtracted from 180 Degrees, leaves for the Supplement $56^{\circ} 52'$, which is the Side DE requir'd.

The Rule which Mr. *Cunn* substitutes in the Room of our Author's, is also universal (but not new); and, consequently, when he says, Change one of the Angles adjacent to the Side sought into its Supplement, it is very just; tho', by the way, I affirm, it is equally true, if the Angle opposite to the Side sought were changed into its Supplement (which perhaps is what has not yet been taken Notice of); only then, instead of having the Side sought directly, we should have its Complement to 180 Degrees, as in the precedent Example; but there is a Necessity of changing either one or all of the Angles into their Supplements, tho' it is best to change only one, which let be either of those next the Side sought, no matter which; and the Side will be had directly without any Subduction, as will appear by the subsequent Operation.

EXAMPLE.

Let the Angle E be changed into its Supplement, and the Side DE sought; which Supplement, and the

other Angle adjacent to the Side sought, being written down first, the Operation may be as follows :

Sup. of the Angle E=50	—	Sine Co. Ar.—	0,115746
The Angle D=30	—	Sine Co. Ar.—	0,301030
Angle A=40	—	Sine 10°—	9,239670
Sum 120	—	Sine 30 —	9,698970
2 —			
$\frac{1}{2}$ Sum 60	—		Sum 19,355416

$\frac{1}{2}$ Sum minus	{	Sup. Angle E=10	—	$\frac{1}{2}$ Sum 9,677708
	{	Angle D=30	—	

Which half Sum 9,677708 gives the Sine of $28^{\circ} 56'$ and the Double thereof $56^{\circ} 52'$ is the Side DE sought the same as before, when all the Angles were changed into their Supplements.

Whence it is abundantly manifest, that those two Methods of Operation, notwithstanding their Manner is so different, agree precisely in Practice; and, consequently, we may conclude our Author's Rule to be right. Wherefore I wonder Mr. *Cunn* did not attend better to the Words of our Author's Rule, before he ventur'd to attack the Characters of so many famous Trigonometrical Writers. But to remove the Imputation of his Charge against those Authors who have deserv'd so well of the Mathematics, and to justify them to the World, (for Justice ought to have place) it is, that I have ventur'd to give my Opinion, and point out where Mr. *Cunn* was mistaken: The Reason of which is not easily assign'd, since, to give him his Due, it could not be for want of Knowledge, tho', in this Case, I can't think it intirely owing to Inadvertency, inasmuch as it was a premeditated Thing; and I am loth to impute it to any contentious Inclinations of his, in disputing the Veracity of our Author's Rule, because it did not appear with all that Plainness requisite to prevent carping by the Litigious: Wherefore, as I am in Suspense how to determine, shall leave the Decision thereof to better Judgments.

Indeed, Mr. *Heynes's* Rule, which directs with the three Angles given to project a Triangle, as if they were Sides, is deficient, were it only on that very Account: For with the given Angles, in the preceding Example,

Example, it will be impossible to construct a Triangle, because 'tis requisite, that two Sides together, however taken, be greater than the third; whereas, in this Case, they will be less: But the Rule is not only deficient in that Respect, but really wrong: For tho' what Mr. *Heynes* asserts is just, viz. that the greatest Side in the supplemental Triangle is the Supplement of the greatest Angle in the other Triangle; yet, notwithstanding that, the Consequence drawn therefrom is false, and so the Solution only imaginary: For, with Submission, neither the Sides, nor their Supplements, in Mr. *Heynes's* supplemental Triangle, are the Measures of the Sides sought. 'Tis true, when one of the Angles is a right one, and the others both acute, then the said Supplemental Triangle is that wanted to be constructed, as containing all the given Angles; and, consequently, the Sides appertaining thereto are the very Sides requir'd: But then this is only one Instance out of the infinite Number of other Triangles that may be constructed, and which is not solv'd directly by the Triangle first projected neither; for the greatest Angle thereof must be changed into its Supplement, when the Side opposite to the right Angle is requir'd; and if the right Angle still remains, and either one or both of the other given Angles are obtuse, the Solution is render'd more perplex'd: Wherefore there can be no general Solution given to any Triangle, by constituting a Triangle whose Sides are equal to the given Angles, except to that particular one which Mr. *Cunn* takes Notice of in his Remark, where each given Angle is the Measure of its opposite Side sought, and which therefore needs no Operation.

This I thought myself oblig'd to observe, out of Justice to Mr. *Cunn*, who we see is not intirely to blame; as having just Reason to object against the Veracity of Mr. *Heynes's* Rule, tho' not against the Rules of the other Authors by him nominated.

And here I can't but take Notice of some Gentlemen, who are so very fond of finding Fault, that, rather than you shall not be in the wrong, they will wrest your own Meaning from you, and will not suffer an Error, tho' ever so minute, to pass, without proclaiming it to the Public, under Pretence of preventing their being impos'd upon; whereas, if the Truth were known, I fear it would appear to be the Vanity of their Hearts,

The APPENDIX.

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Angle } A=40	Sine 10°—9,239670
Sum 120	Sine 30 —9,698970
2 —	
$\frac{1}{2}$ Sum 60	Sum 19,355416

$\frac{1}{2}$ Sum	{ Sup. Angle E=10	—	$\frac{1}{2}$ Sum 9,677708
minus			
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an Over-fondness of being thought wiser and more knowing than the rest of Mankind ; nay, I think it appears plainly so, by their opposing the Works of Men greater than themselves : But if, instead of comparing how far their finite Knowledge extended or exceeded another Person's, they consider'd how much there was they knew nothing of ; as it would conduce to make them humble, so, I am of Opinion, it would contribute very much toward their leaving off that manner of Writing. Besides, as I take it, the Business of Writing is not so much to discover who has committed the most Faults, as to avoid them, and make greater Improvements.

But, what is the most to be wonder'd at, those who are so very ready in finding fault, not without great Suspicion, receive the best Part of their Knowledge from the Works of those very Authors against whom they exclaim. The Reason that induces me to think so is this : Whilst they are studying an Author, in order to understand him, then it is, perhaps, they discover something which he was pleas'd to omit, or thought fit to conceal, for which 'tis more than probable they take care not to omit paying a profound Respect to their vainly imagin'd superior Geniuses : And if, by Accident, an Error should creep in, (which is very possible, none being infallible) then, to be sure, he must be egregiously mistaken, and not understand what he was about : But, I say, this Disquisition into the Demerits of an Author, would never have been made, had they understood the Subject beforehand ; for, if otherwise, they must be of a sad Cynical Temper, as well as have little else to do, to make it their Business to discover Faults, and at the same time acknowledge not one single Beauty ; a very ingrateful Return for the Advantage they receive in the Perusal.

Nor do they do the Public that Service they pretend to ; for those that are capable, and will be at the Trouble, of reading a Treatise upon a Subject without a Master, are as well able as themselves to rectify what is amiss ; and as for those who will not be at that Trouble, there is no Danger of their being led astray ; since it is the same Thing to them, whether there be any Mistakes, or not.

However, if, after all, there should be a Necessity for an Admonition, why can't it be done with Candour and Humanity ?

Humanity? And then, without Doubt, an Author, out of regard to Truth, which of all Things ought to be prefer'd, would be thankful: And to reprove otherwise, is to be ungenerous; because, whenever those Mistakes happen, as they are for the most part owing more to Inadvertency, than want of Knowledge; so they should therefore be attributed to the Frailty of human Nature, (to which we are all more or less subject) nothing being more common amongst all Professions, than the writing of one Thing for another.

If any think, by my interfering between our Author and Mr. *Cunn*, that I have run into the same Error, of which I accuse others in general of being guilty, let them please to consider, that I have only writ in the Vindication of Gentlemen who were first wrongfully accus'd, and in one Particular justify'd Mr. *Cunn*: For such an Occasion as this offering, I thought the Difference between them lay upon me to decide, lest I should be tax'd with Partiality for not doing Justice, or with Ignorance in not determining an Affair which held some in Suspense to know who was in the right or wrong; for there could be no Possibility of making a Merit in adjusting a Thing of so easy a Nature; tho', perhaps, to conceive thoroughly the Reason of all the different Methods of Solution, may not be so easy neither.

But, to proceed: As for the Omissions our Author has made in not determining accurately when some of the Cases are ambiguous, and when not, I shall not quarrel with those who think him to blame; but, if I may be allow'd to give my Opinion, I think they are determin'd, for the most part, as well, or, at least, with more Ease, from the Construction of the Triangles, because it fixes an Idea of what one is about, by exhibiting a kind of an ocular Demonstration; and, consequently, prevents the laying of that Stress upon the Memory as all those are oblig'd to, who depend intirely upon Mr. *Cunn*'s Rules, which to Beginners is not very agreeable: Hence who knows but that what our Author wrote relating to the ambiguous Cases, he thought sufficient? That is, that the Reader would not stop, for want of farther Explications, but with more Ease supply himself with what was wanting when he came to the Practice thereof, I mean the Construction of Triangles (for, after all, without the Knowledge of that, a Person will

have but a mean Notion of this useful Branch of the Mathematics); and, if so, he ought in some measure to be excused, especially if to this we join the following Consideration, *viz.* that few or none ever learn Spherical Trigonometry, purely for the Sake of calculating Sides and Angles, to determine their Ambiguities; besides, what is ambiguous in Trigonometry, is very often not so in Geography and Astronomy, &c. for which the other is chiefly learnt.

For Instance; If we know the Latitude of *London*, and the Distance and Difference of Longitude between the said Place and *Rome*, notwithstanding there are two Sides and an Angle opposite to one of them given, the Case is not doubtful when we undertake to find the Latitude of *Rome*; unless it be not known whether it lies to the *Northward* or *Southward* of *London*; which however could not be determined by any Principles of Trigonometry. Likewise, in Astronomy, if the Latitude of the Place, the Sun's Declination and Azimuth, were given, the *Quæsitum* is not doubtful neither, unless the Sun's Declination exceeds the Latitude of the Place, and both of the same Denomination, that is, both *North*, or both *South*; in which Cases, because it is possible for the Sun to be upon the same Azimuth Circle, twice in the Forenoon; and upon another Azimuth Circle, twice in the Afternoon; it is doubtful, if by Circumstances, during the Observation, we can't discover which of the Times, whether the first or last; but if those Times fall near each other, it will be quite impossible to distinguish which, and therefore ambiguous. Other Instances might be produced, but I believe these are sufficient, to evince that those nice Distinctions are not so necessary in Practice: If there be those who think otherwise, I shall not dispute it, but leave them to their Opinion without Interruption.

However, what with Mr. *Cunn*'s Rules for determining the ambiguous Cases, (which are judiciously drawn up, as including all the Varieties possible) and the Corrections now made by restoring what was lost and corrupted, our Author's Treatise of Trigonometry, in respect to Theory, may perhaps appear complete, even to the most scrupulous. And

Here I thought to conclude; but, for the Sake of Novelty, and to illustrate the various Methods for solving

ing the 12th Case of Oblique Spherics, where the three Angles are given to find either of the Sides, I shall beg leave to give one Instance more, in order to shew how it may be perform'd after a new manner, by the Help of the natural and logarithmical versed Sines; which, if not intirely new, is not so publickly known as the preceding Methods; at least, I never saw any-where the Method of Operation, and therefore shall deliver a Rule for that Purpose in the following Words:

R U L E.

Having, according to the former Directions, chang'd one of the Angles next the Side sought into its Supplement; take the natural versed Sine of the Difference of the said Supplement and the other adjacent Angle, and subtract it from the natural versed Sine of the Angle opposite to the Side sought, and to the Logarithm of the Remainder add the Square of the Radius; then from the Sum subtract the logarithmical Sines of the above Supplement, and the same adjacent Angle; and the Remainder is the Logarithm of a Number, which will be the versed Sine of the Side sought.

E X A M P L E.

Supplement	∠ E	—50°	
Angle	D	—30	
Natural	{	Diff.	—20—,06030
V. Sine	{	∠ A	—40—,23395
			<u> </u>
			,17365
The Log. of which Diff.	17365	{	29,239674
with the Square of Radius is			
Sine of the Sup. of the ∠ E	50°	—	9,884254
Add the Sine of the ∠ D	30	—	9,698970
			<u> </u>
		Sum subtract	19,583224
			<u> </u>
		Remains	9,656450

Which Remainder 9,656450 gives the Logarithm versed Sine of DE 56° 52', agreeing exactly with the former Computations.

Note, If the said Remainder exceeds 10,000000, it implies that the Side sought is greater than a Quadrant; wherefore cancelling the Characteristic 10, look out for the Number answering the remaining Logarithm, from

from which cut off the Left-hand Figure, or, which is the same Thing, abate the Radius (*viz.* Unity); and the Remainder will be the natural Sine of the Excess of the Side sought above a Quadrant.

As the natural and logarithmical versed Sines are not so frequently met with in Books, as the artificial Sines, 'tis possible, on that Account, this Rule may meet with some Objection; for which Reason, and not knowing whether it may be thought preferable to the foregoing Methods, (tho' undoubtedly very easy in Practice) I have omitted its Demonstration; but have publish'd the Rule, with some View of introducing the Use of the former Sines, which sometimes are preferable to the latter: For by the Help of the said versed Sines, and the Reasoning us'd in obtaining this Rule, we necessarily come to the Knowledge of solving that Problem, where two Sides and the contained Angle are given, and the third Side required, at one Operation, very useful in Astronomy and Geography, especially in the latter; when the Latitudes and Longitudes of two Places are given to find their Distance asunder: But the Rule for performing it, and the Demonstration thereof, is also omitted for the sake of Brevity.

However, 'tis easy to perceive, since Angles may be turned into Sides, that the present Rule includes the Solution of that useful Problem in Astronomy, for finding the Sun's Azimuth, having the Latitude of the Place, the Sun's Altitude and Distance from the elevated Pole given; by which means the Variation of the Compasses, of such Importance to Navigators, may be readily determined in any Part of the World.

An Example of which, comprehending the latter Part of the Rule, *viz.* (when the Remainder exceeds 10,000000) is exhibited.

E X A M P L E.

Suppose on *June* the 30th 1732. at *London*, in the Latitude of $51^{\circ} 32'$ No. it were required to find the Sun's true Azimuth, when his Altitude was $50^{\circ} 00'$, in the Afternoon. First,

From

next the Appendix

Pag. 354.

Fig. 1.

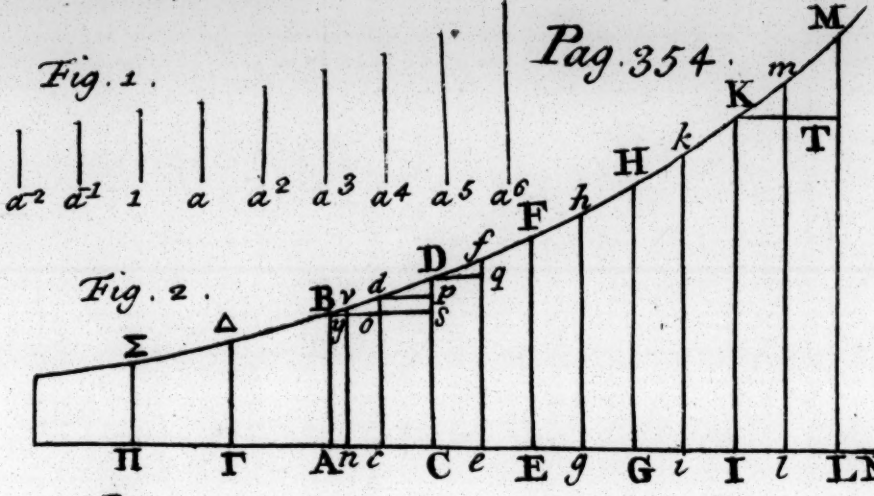


Fig. 2.

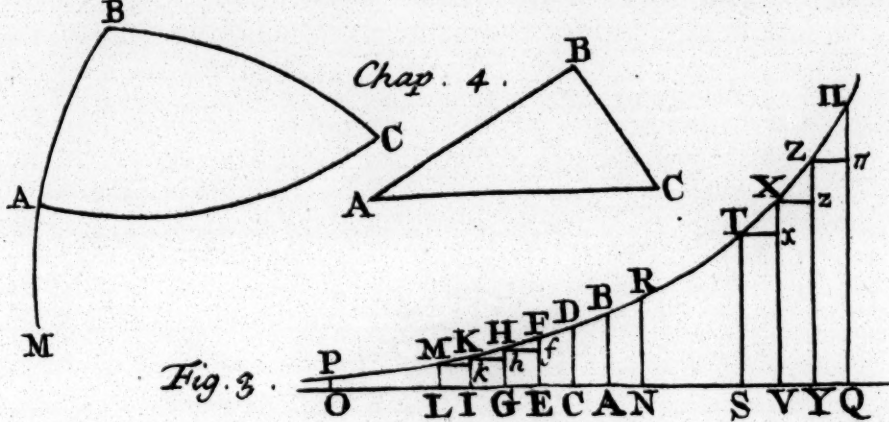
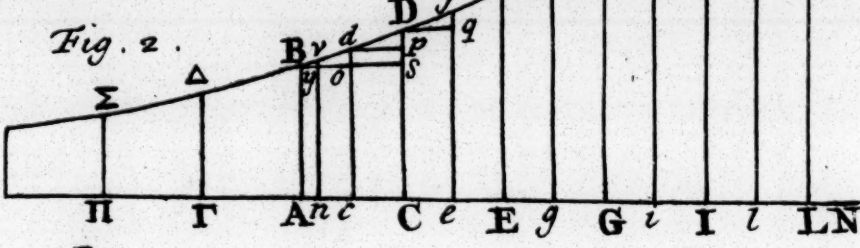


Fig. 4.

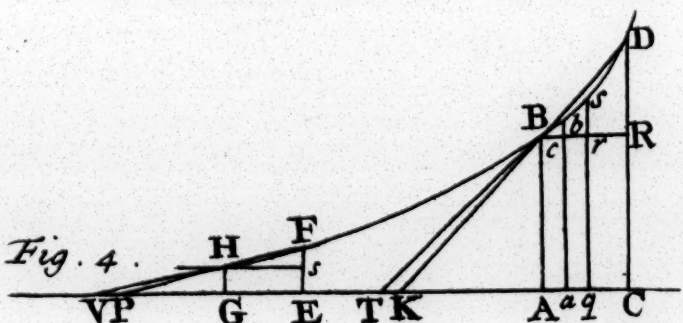
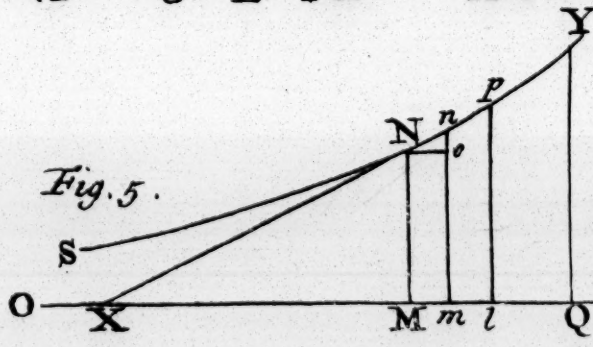
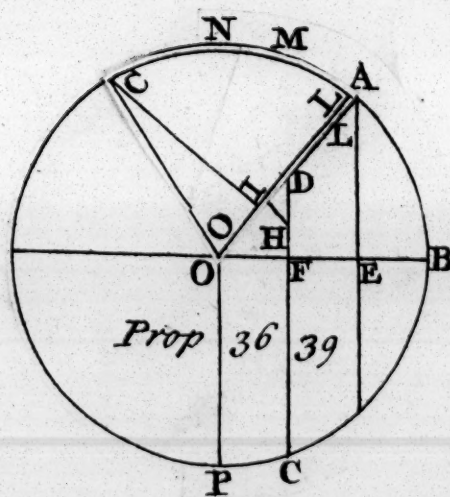
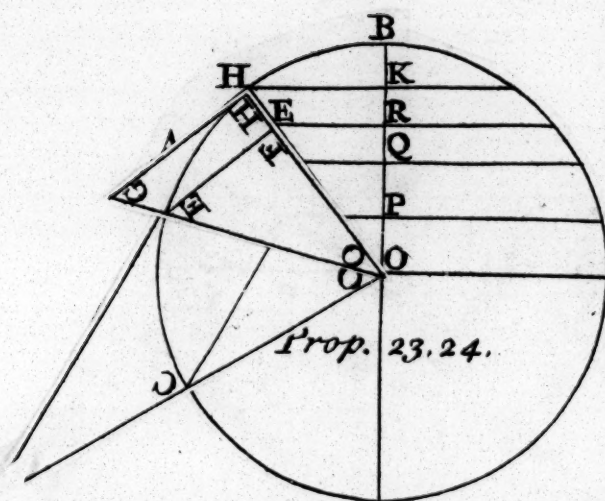
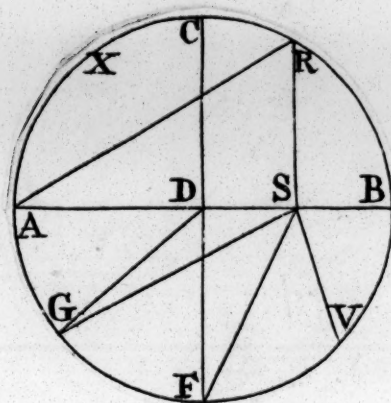


Fig. 5.









From the Com. of the Altitude	0 00
Sub. the Com. of the Latitude	38 28
Natural Sine of the Difference - - -	1 32--00035
V. Sine of the Sun's dist. from the Pole	67 54--62377
	<u>62342</u>
The Log. of which Difference 62342 } with the Square of the Radius, is }	29,794780
Cofine of the { Latitude 51° 32' -	9,793831
Altitude 50 00 -	9,808067
Sum subtract	<u>19,601899</u>
Remains	<u>10,192882</u>

Here the Remainder exceeds 10,000000; wherefore cancel the Characteristic 10, and the Number answering the remaining Logarithm is 1,5591; the Excess of which above Unity, *viz.* .5591, gives the natural Sine of 34° 00'; whence the Sun's true Azimuth is North 124° 00' West: At which Time, if the Sun's Magnetical Azimuth were North 110° 30' West, the Variation of the Compass would be 13° 30' West, as appears by the following Subtraction.

True Azimuth North,	124° 00' West
Mag. Azimuth North,	110 30 West
Variation	<u>13 30 West</u>

N. B. If the Sun's Declination had been South, then the versed Sine of the Sun's Distance from the elevated Pole would have been equal to Unity plus the natural Sine of the Sun's Declination; which in Practice creates no more Trouble than when the Declination is North, if so much; since it is at least as easy to take the natural Sine of an Arc, as to take the versed Sine of its Complement to 90 Degrees; which Sines, and others, with their respective Logarithms, &c. may readily be had out of *Sherwin's* Mathematical Tables.

F I N I S.

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